The Measurement of Intellectual Influence*

Ignacio Palacios-Huerta† and Oscar Volij‡

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Abstract

We examine the problem of measuring influence based on data on the communications between scholarly publications, judicial decisions, patents, web pages, and other entities. The measurement of influence is useful to address several empirical questions such as reputation, prestige, aspects of the diffusion of knowledge, the markets for scientists and scientific publications, the dynamics of innovation, ranking algorithms of search engines in the World Wide Web, and others. In this paper we address the question why any given methodology is reasonable and informative by applying the axiomatic method. We find that a unique ranking method can be characterized by means of five axioms: anonymity, invariance to citation intensity, weak homogeneity, weak consistency, and invariance to splitting of journals. This method is different from those regularly used in social and natural sciences, arts and humanities, and it is at the core of the method used by Google to rank web sites.

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†Department of Economics, Brown University, Providence, RI 02912, USA Email: ipalacios@brown.edu. http:// www.econ.brown.edu/~ iph.

‡Department of Economics, Iowa State University, Ames, IA 50011, USA. Email: oscar@volij.co.il. http://volij.co.il.

1 Introduction

The quality of scientists' research, publications, and achievements plays an important role in the reward and productivity structures within academia. These structures have been the subject of keen interest to economists, sociologists, historians of science, and other scientists. Of particular interest to economists have been the determinants of scholarly productivity, the role of incentives as promoters of the growth of knowledge, and the implications of competition among scientists within the structure of rewards.

Academic journals have played an increasingly important role in both the dissemination of new knowledge and the certification of scientific merit throughout the past century. In an attempt to evaluate the quality of scientific publications, several efforts have been made to measure influence. Citations, as a broad form of influence, are often used in these efforts.

Not only do citations figure prominently in scholarly journals and books, but they also appear in many other forms of documentation, such as patents, newspapers, legal opinions, magazine articles, and in the link structure of the World Wide Web. The literature on citation analysis is by now vast and growing.[1] It is growing mainly because it enables, at a relatively low cost, to make a first attempt at rigorously quantifying elusive but important socioeconomic phenomena such as reputation, influence, prestige, celebrity, the diffusion of knowledge, the quality of scholarly output, the quality of journals, the rise and decline of journals and schools of thought, changes in the publishing process and in the incentives for publication, the returns to publication, the reliance of judicial decisions on previous decisions, and the productivity of scholars, judges, and academic departments.[2] Patents

[1] See Posner (2000) and many references therein.

[2] For example, citations have been used to examine the distribution of influence across journals and changes in this distribution; the extent to which the status of a journal leads to a lengthening of the review process; changes in quality standards and in the distribution of quality in response to increased competition, and other aspects of the publication process (Ellison (2002b)). Citations are also relevant for examining tradeoffs associated with different aspects of paper quality, including the role of social norms for weighing different aspects of quality (Ellison (2002a)); the extent of favoritism on the part of editors (Laband and Piette (1994a)); the incentive for authors to publish, including the monetary value to article publication in terms of direct salary increments, promotion-related salary increments, faculty mobility, and life-cycle productivity profiles (Tuckman and Leahey (1975), Hammermesh, Johnson, and Weisbrod (1982)); the benefits of intellectual collaboration (Laband and Tollison (2000)), and the extent to which academic scientists receive differential returns to publishing articles of varying quality (Sauer (1988)). Citations are also useful to examine the relationship between scholarly significance, academic status, and public fame (Posner (2001)). In the legal profession, citation analysis is extensive. These include studies of judicial citation practices, judicial influence and legal precedent (Landes, Lessig, and Solimine (1998),

also contain citation references to previous patents, which allow one to trace links across inventions. In the literature on the dynamics of innovation and technological change, patent citation data are used for examining the pattern of knowledge spillovers and evaluating the importance of private and governing patenting (see Jaffe and Trajtenberg (2002)). Citation analysis also plays an important role in the World Wide Web in the analysis of "links," where search engines combine sophisticated text matching techniques with a vast link structure to create web search algorithms that find and rank pages according to their importance and relevance to the query. Last, but not least, citation analysis is used widely as a management tool for making decisions on hiring, promotion, salary, and other personnel decisions.

The use of citation analysis is so extensive that *"counting* citations is already a well-established method of empirical research in law, economics, sociology, and academic administration" (Posner (2000), p. 382, italics added). We emphasize the word "counting" because it captures the basis of this methodology. Within economics, for instance, several studies have approximated productivity, quality, and influence by simple citation counts. These counts are then often used to examine various questions of interest such as those mentioned in footnote 2. However, the use of citation counts must be approached with caution. A principal criticism is that the number of citations may be a poor proxy of what is really of interest, whether it is reputation, influence, impact, or the quality and magnitude of a person's achievement.

Some studies attempt to account for this criticism by weighting citations in certain intuitive but ad hoc ways. For instance, the general idea in the evaluation of journals and other scholarly publications is that citations by low impact journals should be given less credit than citations by high impact journals. As a result, there are several arbitrary methods to produce a ranking of journals. For instance, many studies use Garfield's (1972) "impact factor" constructed by the Social Science Citation Index and published by the Institute for Scientific Information, while others follow Liebowitz and Palmer (1984) who proposed various "impact-adjusted" methods for ranking journals. Over the last few

Landes and Posner (1976)), the role of citations both to and in judicial opinions in adjudication and legal decision making, the productivity and influence of judges, the durability of precedents, and the rankings of scholars, books, journals, and schools. See, for instance, the legal citation studies in the conference volume "Interpreting Legal Citations" (*Journal of Legal Studies*, supplement, (2000)).

years, several variations of these methods and others have been applied extensively in many studies in different disciplines.

Given the proliferation of rankings, and the important role that they often seem to play in personnel decisions and in the study of many socioeconomic phenomena of interest, it is somewhat surprising that neither the authors that propose these ranking methods nor those who use them, have tried to justify them.[3] In other words, given that there is, in principle, a plethora of different ranking methods, there is no obvious reason why one should prefer one method over another. Adopting a method because it looks reasonable or because it yields introspectively intuitive results is, to say the least, not the best scientific practice. Without investigating the properties of these methods it is simply not possible to establish a reliably meaningful measure of impact or intellectual influence.

Posner (2000, p. 383) summarizes these criticisms by indicating that "citation analysis is not an inherently economic methodology; most of it has been conducted without any use of the theories or characteristic techniques of economists." The result is that, in terms of quality, the information content of the network of journals, publications, and all other forms of documentation that play a paramount role in the exchange, dissemination, and certification of scientific knowledge is little understood. This, in turn, has implications for all the empirical applications mentioned above.

In this paper we bring economic methodology to bear on the ranking problem. Specifically, we apply the axiomatic method often used in social choice, game theory, and other areas, and present an axiomatic model for measuring intellectual influence.[4] Thus, the approach we take is different from that in the literature. Rather than *assuming* arbitrarily a ranking method on intuitive grounds or introspection, we *derive* a ranking method by requiring a few simple properties. The main result of the analysis is that there is a unique

[3]For instance, Bush, Hamelman, and Staaf (1974), Borokhovich, Bricker, Brunarski, and Simkins (1995), Dusansky and Vernon (1998), Laband and Piette (1994b), Kalaitzidakis, Mamuneas, and Stengos (2001), and many others only describe the method they use, or simply adopt previous methods, and report the resulting rankings. Liebowitz and Palmer (1984), claim that their ranking "probably comes closest to an ideal measure of the impact ... of manuscripts published in various journals," and that their ranking is probably "the closest to 'journal quality'." However, no justification whatsoever is provided for any of these statements.

[4]Arrow's impossibility theorem, Gibbard-Satterthwaite theorem, Nash's derivation of the Nash bargaining solution, Harsanyi's characterization of the utilitarian social welfare function, Peleg's characterization of the core, and Segal's (2000) characterization of the relative utilitarian social preference are just a few examples of the successful application of the axiomatic method.

ranking method that satisfies the proposed properties simultaneously. As it turns out, this method is different from the ones typically used in economics and other sciences to rank journals and departments, and to measure productivity, influence, and prestige.

To focus our discussion we will refer to our entities of interest as "journals." In this context we examine the following properties. The first is the *anonymity* of the method. Roughly, a ranking method is anonymous if it does not depend on the names of the journals under consideration. The second property, which we call *invariance to citation intensity*, requires that the total number of references a journal makes should not affect the ranking of the journals, as long as the distribution of these references does not change. A citing article awards value to the articles it cites. This value is distributed among the cited articles. Thus, the longer the list of the references, the smaller the value awarded to each cited reference. In other words the total value awarded to the articles it cites cannot be increased or decreased by changing the amount of references. The third property concerns the ratio of mutual citations. This property, which we call *weak homogeneity,* is based on Stigler, Stigler, and Friedland (1995), who stress that an important measure of the impact of one journal on another is the ratio of citations of one journal by the other to the citations of the latter to the former.[5] This property requires that in certain two-journal problems, the ratio of the journals' valuations be in a fixed proportion to the ratio of their mutual citations. The fourth property we consider is *weak consistency*, which allows us to extend a ranking method from problems with few journals to problems with more journals. The idea behind this property is that if we know how to rank small problems, we should be able to extend the ranking method to bigger problems in a consistent way. We adopt the notion of consistency that has been applied for axiomatizing rules and solution concepts in diverse problems.[6] Finally, the last axiom, which we call *invariance to splitting of journals*, requires that the splitting of a journal into several identical but smaller journals does not

[5]Stigler, Stigler, and Friedland (1995) also show that an advantage of these ratios is that it is possible to fit a statistical model to the data in terms of simple univariate scores. See also Stigler (1994) for an application of the model to journals in statistics and their relationship with econometrics and economics.

[6]For example, Peleg and Tijs (1996) characterize the Nash equilibrium correspondence, Lensberg (1988) axiomatizes the Nash bargaining solution, Peleg (1985), (1986) characterize the core of NTU and TU games respectively, Hart and Mas-Colell (1989) axiomatize the Shapley value, and Dagan (1994) characterizes the Walrasian correspondence in the context of private ownership economies. In these papers and many others consistency plays a crucial role. Thomson (1990), (2000) offers comprehensive surveys on consistency and its applications.

affect the ranking.

We show that these five properties characterize a *unique* ranking method of journals based on their per-manuscript impact. This method, which we call the Invariant method, was first proposed by Pinski and Narin (1976), and it is different from the methods regularly used to evaluate scholarly publications in economics and in other social and natural sciences.

Our purpose in this paper is not to claim that any given ranking method based on citations is the correct way of measuring impact, much less quality. Citation analysis, however sophisticated it may be, cannot be a substitute for critical reading and expert judging. However, to the extent that the data on the communication between scholarly publications, judicial decisions, patents, web pages, etc., contain valuable information that can be used to address several empirical questions of interest we should ask why a method is reasonable and informative. We thus approach this question by characterizing and comparing different methods according to the properties they satisfy and those they fail to satisfy.

The rest of the paper is organized as follows. Section 2 presents the basic problem of ranking journals according to the average impact of each of its articles and characterizes a ranking method by means of five independent properties. In Section 3 we illustrate the use of these methods with a simple example. Using the citations by articles published in the year 2000 we measure the impact of the articles published during the period 1993-1999 in a set of economics journals. The example shows that differences in the measurement of impact according to different methods can be substantial. Section 4 concludes.

2 The Intellectual Influence of a Publication

In what follows we will use the terms "article" and "journal" to refer to the unit of publication and to the aggregation of these units, respectively. As indicated above, other interpretations can be given to the problem of measuring influence based on communication data depending on the specific application. We thus analyze the problem of ranking journals according to their impact, as measured by the citations their articles generate. We shall be dealing with ranking methods which are functions that take some data as input and return a vector of valuations as output. These valuations should be interpreted as the relative values of the representative articles published in these journals.

2.1 Characterization

Let $\mathcal{J}$ be a nonempty set of journals. This set is to be interpreted as the universe of all potential journals. For each $i \in \mathcal{J}$ and for each $t_i \in I\!N$, we say that (i, t_i) is a replica of i. We assume that $\mathcal{J}$ is closed under replication. That is, if i belongs to $\mathcal{J}$ then so do all its replicas. Let $J \subseteq \mathcal{J}$ be a finite subset of journals. A *citation matrix* for J is a $|J| \times |J|$ non-negative matrix (c_{ij}). For each $i, j \in J$, c_{ij} represents the number of *citations* to journal i by journal j, or the number of *references* of journal j to journal i. For $j \in J$, we denote by $\bar{c}_j$ the vector $(c_{ij})_{i \in J}$ of journal j's references, and the sum of all journal j's references is denoted by c_j, namely $c_j = \sum_{i \in J} c_{ij}$. All vectors are *column* vectors. For a vector v, $\|v\|$ denotes the 1-norm of v, namely $\|v\| = \sum_{i \in J} |v_i|$. The diagonal matrix with $d_1, \ldots, d_n$ as its main diagonal entries is denoted by $\text{diag}(d_1, \ldots, d_n)$. Given a matrix of citations $C = (c_{ij})$, we define $D_C = \text{diag}(c_j)_{j \in J}$ to be the diagonal matrix with the sums of the journals' references as its main diagonal. Further, the matrix CD_C^{-1} will be called the normalized matrix of C and it is readily seen to be a stochastic matrix (the entries of each of its columns add up to one). When the matrix of citations C is understood from the context, we shall write D instead of D_C. This should cause no confusion.

Given a matrix of citations C for J, we say that journal i *is cited* by journal j if $c_{ij} > 0$. We say that journal i *impacts* journal j if there is a finite sequence $i_0, \ldots i_n$, with $i_0 = i$ and $i_n = j$, such that for all $t = 1, \ldots, n$, journal i_{t-1} is cited by journal i_t. Journals i and j *communicate* if either $i = j$ or if they impact each other. It is easy to see that the communication relation is an equivalence relation and, therefore, it partitions the set J of journals into equivalence classes, which we call communication classes. A *discipline* is a communication class $J' \subseteq J$ such that no journal in $J \setminus J'$ impacts any journal inside J'. If a matrix of citations C has two disciplines, this means that there are two communication classes in J that are disconnected. Namely, there is no chain of citations that go from a journal in one discipline to a journal in another and vice versa. Since we are interested in rankings within a single discipline, we will restrict attention to citation matrices whose set of journals constitute a single discipline.[7] This leads to the following definition:

[7]Such matrices are known as non-negative, irreducible matrices.

Definition 1 A *ranking problem* is a triple $\langle J, a, C \rangle$ where $J \subseteq \mathcal{J}$ is a set of journals, $a = (a_i)_{i \in J}$ is the vector of number of articles they published, and $C = (c_{ij})_{(i,j) \in J \times J}$ is a citation matrix for J with J as its only one discipline.

The primitives of a ranking problem consist of the relevant set of journals, the number of articles in each journal, and the corresponding matrix of citations. Clearly, the choice of the relevant set of journals will generally affect the results of the implementation of a ranking method. Thus, it is important to make a good choice of journals. In our analysis, however, we take as given the set of journals, and just deal with the problem of measuring influence within this set.

There is a class of problems that will play an important role in our analysis, those in which every journal has the same number of articles, i.e. $a_i = a_j$ for all $i, j \in J$. These problems will be called *isoarticle problems*, and will also be helpful for illustration purposes.

We are interested in building a cardinal ranking of the journals in J, namely a non-zero vector of non-negative valuations $(v_j)_{j \in J}$. Each v_j is to be interpreted as the value of a representative article in journal i. Since only relative values matter, we can normalize the vector of valuations so that they add up to 1. Denote the set of all possible vectors of valuations of J by Δ_J. That is, $\Delta_J = \{(v_j)_{j \in J} : v_j \geq 0, \sum_{j \in J} v_j = 1\}$. Further, $\Delta = \cup_{J \subseteq \mathcal{J}} \Delta_J$.

Definition 2 Let $\mathcal{R}$ be the set of all ranking problems. A *ranking method* is a function $\phi : \mathcal{R} \to \Delta$, that assigns to each ranking problem $\langle J, a, C \rangle$ a vector of valuations $v \in \Delta_J$.

Given a vector $(a_i)_{i \in J}$ of number of articles, we will denote by A the diagonal matrix $\text{diag}(a_i)_{i \in J}$. The following are some examples of different ranking methods.

Examples:

1. The Egalitarian method is the function that assigns the same value to every journal. Formally, $\phi_E : \mathcal{R} \to \Delta$ is defined by $\phi_E(J, a, C) = (1/|J|, \ldots, 1/|J|)^T$.

2. The Counting method awards each journal the proportion of its citations out of the total number of citations. Formally: $\phi_C : \mathcal{R} \to \Delta$ is defined by $\phi_C(J, a, C) = (\frac{\sum_{j \in J} c_{ij}/a_i}{\sum_{j \in J} c_j/a_j})_{i \in J}$.

3. The Modified Counting method awards each journal the proportion of its non-self-citations out of the total number of non-self-citations. Formally: $\phi_{MC} : \mathcal{R} \to \Delta$ is defined by $\phi_{MC}(J, a, C) = (\frac{\sum_{j \in J \setminus \{i\}} c_{ij}/a_i}{\sum_{k \in J} \sum_{j \in J \setminus \{k\}} c_{kj}/a_k})_{i \in J}$.

4. The Liebowitz-Palmer method $\phi_{LP} : \mathcal{R} \to \Delta$ assigns to each ranking problem $R = \langle J, a, C \rangle$ the unique fixed point of the operator $T : \Delta_J \to \Delta_J$ defined by $T(v) = \frac{A^{-1}Cv}{\|A^{-1}Cv\|}$.

5. The Invariant method ϕ_I assigns to each ranking problem $R = \langle J, a, C \rangle$, the unique member of $v \in \Delta_J$ that satisfies $CD_C^{-1}Av = Av$.

The Counting method was first used in Bush, Hamelman, and Staaf (1974).[8] This method was later used by Liebowitz and Palmer (1984) (see their Table 2, column 2 (Rankings Based on Citations Per Article)). The Modified Counting method was used in Bush, Hamelman, and Staaf (1974) (see their Table 1, column "Diagonal Excluded"). As to the Liebowitz-Palmer method, it was proposed by Liebowitz and Palmer (1984) and used in the construction of the journal ranking that appears in their Table 2, column 4 (Rankings Based on Impact Adjusted Citations Per Article). The Invariant method was first proposed by Pinski and Narin (1976) (see also Daniels (1969) and Moon and Pullman (1970) for an earlier definition in the context of tournaments). They build three related influence measures, and their second one, namely their influence per publication, coincides with the Invariant method. They define this measure in a different way from ours, but Geller (1978) later showed that Pinski and Narin's influence per publication is equivalent to the Invariant method. The Invariant method is at the core of the method used by

[8] Although these authors do not assign numerical values to each of the journals they consider, the overall journal ranking in their Table 1, column "Diagonal Included," is built according to the Counting method.

Google to rank web pages, known as PageRank (see for example Page, Brin, Motwani, and Winograd (1999)). To the best of our knowledge, the Egalitarian method has never been used for ranking journals or other publications in any discipline.

The reader may wonder whether the Liebowitz-Palmer and the Invariant methods are well-defined. After all, the operator that is used to define the Liebowitz-Palmer method may have more than one fixed point, and the equation that defines the Invariant method may not have a unique solution. That the Invariant method is well-defined follows from the fact that the normalized matrix CD^{-1} is an irreducible stochastic matrix and that every irreducible stochastic matrix has a unique invariant distribution. The fact that the Liebowitz-Palmer method is well defined is a corollary of the Perron-Frobenius theorem for irreducible matrices. Specifically, note that $\phi_{LP}(J, a, C)$ is a characteristic vector in Δ_J of the non-negative and irreducible matrix $A^{-1}C$.[9] One of the results of the Perron-Frobenius theory is that every irreducible, non-negative, $|J| \times |J|$ matrix M has exactly one eigenvector in Δ_J (see Minc (1988), Theorem 4.4). Therefore, the Liebowitz-Palmer method can be defined alternatively as assigning to each ranking problem $\langle J, a, C\rangle$ the only eigenvector of $A^{-1}C$ in Δ_J. Incidentally, what Liebowitz and Palmer (1984), Laband and Piette (1994b), and other papers that use the Liebowitz-Palmer method actually do is to calculate the eigenvector by means of an iterative procedure known as the Power method. For a discussion of this procedure see Wilkinson (1965).

It is useful to compare the Invariant and the Liebowitz-Palmer methods. Both of them calculate a positive eigenvector of an appropriately adjusted matrix of citations. The Liebowitz-Palmer method calculates the positive eigenvector of the matrix $A^{-1}C$, while the Invariant method calculates the positive eigenvector of the matrix $A^{-1}CD^{-1}A$. The entry c_{ij}/a_i of the matrix $A^{-1}C$ is the average number of citations that an article in journal i gets from journal j. This is the underlying measure of direct impact (of a typical article in i on a typical article in j) that the Liebowitz-Palmer method takes into account. The Invariant method controls for citation intensity by dividing the value c_{ij}/a_i by $\sum_k c_{kj}/a_j$, that is, by the average length of the list of references of the articles in j. Therefore, the measure of direct impact of journal i on journal j that underlies the Invariant method is the average number of citations of an article in i out of the average number of citations by

[9]That is, $\phi_{LP}(J, a, P)$ is a non-zero vector that solves $A^{-1}Cv = \lambda v$ for some λ.

a typical article of j.

Note that both the Liebowitz-Palmer and the Invariant methods assign to journal i a value that is a weighted average of some function of the citations it gets: $v_i = \sum_{j\in J} \alpha_{ij} v_j$. For the Invariant method, $\alpha_{ij} = \frac{c_{ij}/a_i}{c_j/a_j}$, while for the Liebowitz-Palmer method, $\alpha_{ij} = \frac{c_{ij}/a_i}{\|Cv\|}$. According to these measures, not all citations have the same value. Citations by important journals are more valuable than citations by less important journals. But the importance of a journal is determined endogenously and simultaneously with the importance of all other journals.

It is clear that one can think of a large number of ranking methods, the above examples being just a few. In applications, it would be desirable to choose an appropriate one. What is appropriate, however, depends on the different properties that the ranking method may satisfy. Therefore, in order to analyze and distinguish among different methods, we should evaluate the properties that each one satisfies. Next, we follow this approach and characterize a ranking method by means of some basic, desirable properties.

The first property we consider is *anonymity.* It says that the ranking of a set of journals should not depend on the names of the journals. Recall that a permutation matrix is a (0,1)-matrix that has exactly one 1 in each row and each column.

Definition 3 A ranking method ϕ satisfies *anonymity* if for all ranking problems $R = \langle J, a, C\rangle$ and for all $|J| \times |J|$ permutation matrices P, $\phi(J, Pa, PCP^T) = P\phi(J, a, C)$.

The following diagram illustrates this property for isoarticle problems with two journals:

$$\text{If} \quad \begin{matrix} & i & j \\ i & \multicolumn{2}{c}{} \\ j & & \end{matrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} v_i \\ v_j \end{pmatrix} \quad \text{then} \quad \begin{pmatrix} d & c \\ b & a \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} v_j \\ v_i \end{pmatrix}.$$

That is, a permutation of the matrix of citations results in the same permutation of the valuations.

In order to motivate the next property, consider a ranking problem $\langle J, a, C\rangle$ where for each journal $j \in J$, journal j's list of references is given by the vector $\bar{c}_j = (c_{ij})_{i\in J}$. The vector $\bar{c}_j$ represents journal j's opinions about the journals in J. These opinions are given by the ratios c_{ij}/c_{kj} of j's references to the different journals. These opinions do not change if journal j were to modify the number of references by multiplying then by a constant $\lambda_j > 0$, thus turning the vector $\bar{c}_j$ into the vector $\lambda_j \bar{c}_j$. The second property requires that the ranking method not be affected by such changes. If there is no change in the journal's opinions, the ranking should not change. In other words, all else equal, the length of the reference section should not matter.

Definition 4 A ranking function ϕ satisfies *invariance with respect to citation intensity* if for every ranking problem $\langle J, a, C \rangle$ and for every diagonal matrix $\Lambda = \operatorname{diag}(\lambda_j)_{j \in J}$ with strictly positive diagonal entries, $\phi(J, a, C\Lambda) = \phi(J, a, C)$.

The following diagram illustrates this property for isoarticle ranking problems:

$$\text{if} \quad \begin{matrix} & i & j \\ i & a & b \\ j & c & d \end{matrix} \ \left(\begin{matrix} a & b \\ c & d \end{matrix}\right) \xrightarrow{\phi} \left(\begin{matrix} v_i \\ v_j \end{matrix}\right) \quad \text{then} \quad \left(\begin{matrix} \lambda_i a & \lambda_j b \\ \lambda_i c & \lambda_j d \end{matrix}\right) \xrightarrow{\phi} \left(\begin{matrix} v_i \\ v_j \end{matrix}\right).$$

The idea behind this property is that, given the content of a journal, each article should have one vote. If an article in journal j refers to many articles in different journals, then that article's vote is divided among the cited journals. Making an analogy, given the content of a paper, the value of a paper is distributed among its coauthors. All else equal, the greater the number of coauthors, the smaller the credit each one gets. That is, the value of a paper cannot be modified by leaving intact its content and changing the number of authors.

Recall that given a matrix of citations C, the matrix D of number of cited references has the total number of citations by each of the journals as its main diagonal. Therefore, the matrix $A^{-1}D$ has the journals' citation intensities, $(c_i/a_i)_{i \in J}$, as its main diagonal. The next two properties concern problems where all journals have the same citation intensity, namely where $A^{-1}D = \kappa I$ for some $\kappa > 0$. Such problems will be called *homogeneous*. The next property we examine is motivated by the analysis in Stigler, Stigler, and Friedland (1995).

Definition 5 The ranking function ϕ satisfies *weak homogeneity* if there is $\alpha > 0$ (that may depend on $\{i, j\}$ but not on a nor C), such that for all homogeneous and isoarticle two-journal problems $R = \langle \{i, j\}, a, C\rangle$, $\phi_i(R)/\phi_j(R) = \alpha c_{ij}/c_{ji}$. We say that ϕ satisfies *homogeneity* if the above condition holds for all homogeneous two-journal problems, not necessarily isoarticle ones.

This property says that in two-journal problems where both journals have the same number of articles and the same citation intensity, the relative valuation of a journal should be proportional to the ratio of their mutual citations.

The following diagram illustrates this property:

$$\begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cc} \quad i \quad & \quad j \quad \end{array} \\ \begin{pmatrix} K - c_{ji} & c_{ij} \\ c_{ji} & K - c_{ij} \end{pmatrix} \end{array} \xrightarrow{\phi} \begin{pmatrix} v_i \\ v_j \end{pmatrix} = \frac{1}{c_{ji} + \alpha c_{ij}} \begin{pmatrix} \alpha c_{ij} \\ c_{ji} \end{pmatrix}.$$

The value c_{ij} is a measure of i's direct influence on j. Thus, the ratio c_{ij}/c_{ji} represents the direct influence of journal i on journal j relative to the direct influence of journal j on journal i. The importance of these ratios was stressed in Stigler, Stigler, and Friedland (1995), where these sender-receiver ratios, as they call them, were calculated for a group of nine core journals. In building a desirable ranking method, however, one would like to take into account not only the direct influence of each journal on each of the others, but also the indirect influence. This is why these ratios, though conveying important information, are not, per se, a perfect index of the journals' total impact. In a two-journal problem, however, the value c_{ij} *is* a measure of the total impact of journal i on journal j. Stigler, Stigler, and Friedland (1995) admit that "[t]hese sender-receiver ratios are influenced by the varying number of citations in articles published by each journal." They are not clear, however, as to whether that variation calls for correction. In any case, our weak homogeneity property requires that the ratio of valuations be proportional to the ratio of mutual citations *only* in two-journal isoarticle problems where the citation intensity of each journal is the same. In this way, the varying number of citations and the effects of indirect influences across journals are not an issue.

The fourth property will allow us to relate large problems to small problems. Thanks to this property, if we know how to solve a ranking problem with few journals, we will also know how to solve problems with a greater number of journals. The idea is to extend a ranking method of few journals to a ranking method of more journals in a consistent way. In order to formalize what we mean by *consistency,* we first need some definitions.

Let $R = \langle J, (a_i)_{i\in J}(c_{ij})_{(i,j)\in J\times J}\rangle$ be a ranking problem, and let $k \in J$. The *reduced ranking problem with respect to* k is $R^k = \langle J \setminus \{k\}, (a_i)_{i\in J\setminus\{k\}}, (c^k_{ij})_{(i,j)\in J\setminus\{k\}\times J\setminus\{k\}}\rangle$, where:

$$c^k_{ij} = c_{ij} + c_{kj}\frac{c_{ik}}{\sum_{t\in J\setminus\{k\}} c_{tk}} \qquad \text{for all } i, j \in J \setminus \{k\}.$$

Note that since $(c_{ij})_{(i,j)\in J\times J}$ is irreducible, $\sum_{t\in J\setminus\{k\}} c_{tk} > 0$, and hence, R^k is well-defined. Further, $(c^k_{ij})_{(i,j)\in J\setminus\{k\}\times J\setminus\{k\}}$ is itself irreducible.

The reduced problem represents the following situation. Suppose we want to rank the journals in J and our computer cannot deal with $|J| \times |J|$ matrices but only with $(|J| - 1) \times (|J| - 1)$ matrices. Therefore we need to resize our problem and abstract from one journal in our data set, say journal k. Still, we are interested in the relative values of all the remaining journals. If we eliminated journal k from the matrix, namely if we eliminated the corresponding row and column, we would lose some valuable information. Therefore, we need to "touch up" the matrix so that the information of the missing journal is not lost. In the old matrix, c_{kj} was the number of citations to journal k by journal j. If we do not want to lose this information, we need to redistribute these citations among the other journals; the appropriate way to do so is in proportion to the citations (or opinions) by the missing journal k. In other words, journal j gave credit to journal k in the form of c_{kj} citations while journal k gave credit to the journals other than k according to the vector $\bar{c}_k$ of k's references. Therefore, if we do not want to lose the information about the indirect impact of each of the journals in $J \setminus \{k\}$ on $J \setminus \{k\}$, we need to redistribute each of the c_{kj} citations, $j \in J \setminus \{k\}$, according to the relative number of k's references to the other journals; that is in proportion to the values in $\bar{c}_k$. If this is the correct method to recover the information lost by the need to use a smaller matrix, we should expect that our ranking method give, at least in homogeneous problems, the same relative valuations to the journals in $J \setminus \{k\}$ both when applied either to the original problem and to the reduced one. This is the requirement of the next property.

Definition 6 The ranking function ϕ satisfies *weak consistency* if for all homogeneous, isoarticle problems $R = \langle J, a, C\rangle$, with $|J| > 2$, and for all $k \in J$,

$$\frac{\phi_i(R)}{\phi_j(R)} = \frac{\phi_i(R^k)}{\phi_j(R^k)} \qquad \text{for all } i, j \in J \setminus \{k\}.$$

We say that ϕ satisfies *consistency* if the above condition holds for all homogeneous problems, not necessarily isoarticle ones.

The property of consistency requires from a ranking method that the relative valuations of the journals of a homogeneous problem not be affected if we apply the method to the reduced problem with respect to k.

The following diagram illustrates the consistency requirement for an isoarticle problem:

$$\text{If} \quad \begin{array}{c} \\ i \\ j \\ k \end{array} \begin{array}{c} \begin{array}{ccc} i & j & k \end{array} \\ \begin{pmatrix} 30 & 16 & 10 \\ 15 & 35 & 20 \\ 15 & 9 & 30 \end{pmatrix} \end{array} \xrightarrow{\phi} \begin{pmatrix} v_i \\ v_j \\ v_k \end{pmatrix} \quad \text{then} \quad \begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} 30+5 & 16+3 \\ 15+10 & 35+6 \end{pmatrix} \end{array} \xrightarrow{\phi} \begin{pmatrix} \frac{v_i}{v_i+v_j} \\ \frac{v_j}{v_i+v_j} \end{pmatrix}.$$

Note that the 15 citations from journal i to journal k are distributed between i and j in the proportion $10/20 = c_{ik}/c_{jk}$ of the citations received from k. Similarly, the 9 citations from j to k are distributed among i and j in the same proportions.

The properties introduced so far, are sufficent to rank journals according to their *overall* impact. The next and last property of ranking methods will enable us to rank journals with different number of articles, according to their *per-manuscript impact*. In order to motivate it, consider a ranking problem $\langle J, a, C\rangle$ and suppose that a journal $j \in J$ splits into two identical journals. Specifically, j splits into journal $(j, 1)$ and journal $(j, 2)$, each with $a_j/2$ articles. Further, for these two new journals to be equivalent, the citations are also split: the vectors of citations by journals $(j, 1)$ and $(j, 2)$ to the other journals are equal and given by $(c_{ij}/2)_{i\neq j}$. Also, the citations of journal j are equally split: each journal $i \neq j$ cites $c_{ji}/2$ times each of the newly-born journals. Lastly, the self-citations c_{jj} are equally split between the two journals, giving $c_{(j,\alpha),(j,\beta)} = c_{jj}/4$ for $\alpha, \beta = 1, 2$. For a case of a two-journal problem, this split is illustrated as follows:

$$\left\langle \begin{pmatrix} a_i \\ a_j \end{pmatrix}, \begin{array}{c} \\ i \\ j \end{array} \begin{array}{c} \begin{array}{cc} i & j \end{array} \\ \begin{pmatrix} c_{ii} & c_{ij} \\ c_{ji} & c_{jj} \end{pmatrix} \end{array} \right\rangle \longrightarrow \left\langle \begin{pmatrix} a_i \\ a_j/2 \\ a_j/2 \end{pmatrix}, \begin{array}{c} \\ i \\ (j,1) \\ (j,2) \end{array} \begin{array}{c} \begin{array}{ccc} i & (j,1) & (j,2) \end{array} \\ \begin{pmatrix} c_{ii} & c_{ij}/2 & c_{ij}/2 \\ c_{ji}/2 & c_{jj}/4 & c_{jj}/4 \\ c_{ji}/2 & c_{jj}/4 & c_{jj}/4 \end{pmatrix} \end{array} \right\rangle.$$

We would like this split of journal j not to influence the ranking of the articles published. In other words, the old and the new relative rankings should be the same. We formalize this property next.

Let $R = \langle J, (a_j)_{j\in J}, (c_{ij})_{(i,j)\in J\times J}\rangle$ be a ranking problem. Each journal $j \in J$ will be split into $T_j \geq 1$ identical journal, denoted (j, t_j), for $t_j = 1, \ldots T_j$. With some abuse of notation, we shall denote by T_j both the number and the set of "types" of journal j. The resulting ranking problem is $R' = \langle J', (a'_{j,t_j})_{j\in J, t_j \in T_j}, (c'_{(i,t_i)(j,t_j)})_{((i,t_i),(j,t_j))\in J'\times J'}\rangle$ where $J' = \{(j, t_j) : j \in J, t_j \in T_j\}$, $a'_{j,t_j} = a_j/T_j$ and $c'_{(i,t_i)(j,t_j)} = \frac{c_{ij}}{T_i T_j}$. We will call the problem R' a split of R, and we will denote its citation matrix $(c'_{(i,t_i)(j,t_j)})_{((i,t_i),(j,t_j))\in J'\times J'}$ by C'.

As mentioned above, we would expect a split of a journal not to affect the relative valuations of the articles. This is the requirement imposed by the following property.

Definition 7 A ranking method ϕ satisfies *invariance to splitting of journals* if for all ranking problems $R = \langle J, (a_j)_{j\in J}, (c_{ij})_{(i,j)\in J\times J}\rangle$, for all $i, j \in J$ and for all its splittings $R' = \langle J', (a'_{(j,t_j)})_{j\in J, t_j\in T_j}, (c'_{(i,t_i)(j,t_j)})_{((i,t_i),(j,t_j))\in J'\times J'}\rangle$, we have:

$$\phi_i(R)/\phi_j(R) = \phi_{(i,t_i)}(R')/\phi_{(j,t_j)}(R') \qquad \forall i, j \in J \text{ and } \forall t_i \in T_i \text{ and } t_j \in T_j.$$

We are now ready to characterize the only ranking method that satisfies all the properties described above.

Theorem 1 There is a unique ranking method that satisfies anonymity, invariance to citation intensity, weak homogeneity, weak consistency, and invariance to splitting of journals. It is the Invariant method ϕ_I.

Proof : We first show that the Invariant method satisfies the stated properties. Later we will show that no other method satisfies all these properties simultaneously.

The Invariant method satisfies invariance to citation intensity since for any citation matrix C and for any diagonal matrix Λ with positive diagonal entries, C and $C\Lambda$ have the same normalized matrix. To see that this method satisfies anonymity, let P be a permutation matrix and assume that v^* solves $A^{-1}CD^{-1}Av = v$. We need to show that Pv^* solves $(PA^{-1}P^T)(PCP^T)(PD^{-1}P^T)(PAP^T)v = v$. But if $v^* = A^{-1}CD^{-1}Av^*$, then

$$\begin{aligned} Pv^* &= PA^{-1}CD^{-1}Av^* \\ &= (PA^{-1}P^T)(PCP^T)(PD^{-1}P^T)(PAP^T)Pv^*. \end{aligned}$$

To see that it satisfies homogeneity, and a fortiori weak homogeneity, let $R = \langle J, a, C\rangle$ be a two-journal problem such that $A^{-1}D = \kappa I$. That is, $D = \kappa \operatorname{diag}(a_1, a_2)$ and

$$C = \begin{pmatrix} \kappa a_1 - c_{21} & c_{12} \\ c_{21} & \kappa a_2 - c_{12} \end{pmatrix}.$$

But then,

$$\begin{aligned} A^{-1}CD^{-1}A(c_{12}, c_{21})^T &= \frac{1}{\kappa}A^{-1}C(c_{12}, c_{21})^T \\ &= D^{-1}C(c_{12}, c_{21})^T \\ &= (c_{12}, c_{21})^T. \end{aligned}$$

That is, $\phi_I(R) = \frac{(c_{12},c_{21})^T}{c_{12}+c_{21}}$, and the result follows.

Let us next show that ϕ_I also satisfies consistency (and, a fortiori, weak consistency). Let $R = \langle J, a, C\rangle$ be a homogeneous problem and let $(v_i^*)_{i\in J} = \phi_I(R)$. That is $(v_i^*)_{i\in J}$ solves

$$A^{-1}CD^{-1}Av = v. \qquad (1)$$

Since $DA^{-1} = \kappa I$ and $D^{-1}A = 1/\kappa I$, pre-multiplying both sides of (1) by D we have that

$$Cv^* = Dv^*.$$

That is,

$$\sum_{j\in J} c_{ij}v_j^* = v_i^* c_i \qquad \text{for all } i \in J. \tag{2}$$

Now let $k \in J$ and let $R^k = \langle J \setminus \{k\}, (a_i)_{i\in J\setminus\{k\}}, C^k\rangle$ be the reduced problem with respect to k. Note that since R is a homogeneous problem, so is R^k. Further, $c_j^k = \sum_{i\in J\setminus\{k\}} c_{ij}^k = \sum_{i\in J} c_{ij} = c_j$ for all $j \in J \setminus \{k\}$. Denoting $(A|k) = \text{diag}(a_i)_{i\in J\setminus\{k\}}$ and $(D|k) = \text{diag}(c_i)_{i\in J\setminus\{k\}}$, we have that $\phi(R^k)$ solves $(A|k)^{-1}C^k(D|k)^{-1}(A|k)v = v$, or, pre-multiplying both sides by $(D|k)$, or

$$C^k v = (D|k)v.$$

It is enough to show that $(v_i^*)_{i\in J\setminus\{k\}}$ satisfies the above equation, namely

$$\sum_{j\in J\setminus\{k\}} c_{ij}^k v_j^* = v_i^* c_i \qquad \text{for all } i \in J \setminus \{k\}.$$

By definition of c_{ij}^k,

$$\begin{aligned}
\sum_{j\in J\setminus\{k\}} c_{ij}^k v_j^* &= \sum_{j\in J\setminus\{k\}} (c_{ij} + c_{kj}\frac{c_{ik}}{\sum_{t\in J\setminus\{k\}} c_{tk}})v_j^* \\
&= \sum_{j\in J\setminus\{k\}} c_{ij}v_j^* + \sum_{j\in J\setminus\{k\}} (c_{kj}\frac{c_{ik}}{c_k - c_{kk}})v_j^*.
\end{aligned}$$

Since by (2), $v_i^* c_i = \sum_{j\in J} c_{ij}v_j^*$, we have $\sum_{j\in J\setminus\{k\}} c_{ij}v_j^* = v_i^* c_i - c_{ik}v_k^*$. Therefore,

$$\begin{aligned}
\sum_{j\in J\setminus\{k\}} c_{ij}^k v_j^* &= v_i^* c_i - c_{ik}v_k^* + \frac{c_{ik}}{c_k - c_{kk}} \sum_{j\in J\setminus\{k\}} c_{kj}v_j^* \\
&= v_i^* c_i - c_{ik}v_k^* + \frac{c_{ik}}{c_k - c_{kk}}(v_k^* c_k - c_{kk}v_k^*) \\
&= v_i^* c_i - c_{ik}v_k^* + \frac{c_{ik}}{c_k - c_{kk}}v_k^*(c_k - c_{kk}) \\
&= v_i^* c_i.
\end{aligned}$$

Finally, we shall show that the Invariant method satisfies invariance to splitting of journals. Let $R = \langle J, a, C\rangle$ be a ranking problem and let $R' = \langle J', a', C'\rangle$ be a splitting of R, where $J' = \{(j, t_j) : j \in J, t_j \in T_j\}$, $a'_{(j,t_j)} = a_j/T_j$ and $c'_{(i,t_i)(j,t_j)} = \frac{c_{ij}}{T_i T_j}$. Let

$(v_i)_{i\in J} = \phi_I(R)$ and $(v_{(i,t_i)})_{(i,t_i)\in J\times T_i} = \phi_I(R')$. By definition of $\phi_I(R)$ we know that

$$v_i = \sum_{j\in J} \frac{c_{ij}}{c_j}\frac{a_j}{a_i} v_j, \qquad \text{for all } i \in J.$$

For each $j \in J$, choose $t_j \in T_j$. We need to show that $v_{(i,t_i)}/v_{(j,t_j)} = v_i/v_j$ for all $i, j \in J$. It is enough to show that

$$v_{(i,t_i)} = \sum_{j\in J} \frac{c_{ij}}{c_j}\frac{a_j}{a_i} v_{(j,t_j)}, \qquad \text{for all } i \in J.$$

By definition of $\phi_I(R')$ we have

$$\begin{aligned} v_{(i,t_i)} &= \sum_{j\in J}\sum_{t_j\in T_j} \frac{c'_{(i,t_i)(j,t_j)}}{c'_{(j,t_j)}}\frac{a'_{(j,t_j)}}{a'_{(i,t_i)}} v_{(j,t_j)} \\ &= \sum_{j\in J}\sum_{t_j\in T_j} \frac{\frac{c_{ij}}{T_i T_j}}{c_j/T_j}\frac{a_j/T_j}{a_i/T_i} v_{(j,t_j)} \\ &= \sum_{j\in J} \frac{c_{ij}}{a_i}\frac{a_j}{c_j} v_{(j,t_j)}, \end{aligned}$$

which is what we wanted to prove.

We shall show next that a ranking method that satisfies the five axioms must be the Invariant method. The method of proof is the following. We first show that any ranking method that satisfies anonymity, invariance to citation intensity, and weak homogeneity must coincide with the Invariant method in two-journal isoarticle problems. After that, we use weak consistency to extend this result to all isoarticle problems. Lastly, we use invariance to splitting of journals to extend the result to all ranking problems.

Lemma 1 Let $\phi : \mathcal{R} \to \Delta$ be a ranking method that satisfies anonymity, invariance to citation intensity, and weak homogeneity. Then, for every two-journal, isoarticle problem $R = \langle \{i, j\}, a, C\rangle$, $\phi(R) = \phi_I(R)$.

Proof : Let $R = \langle \{i, j\}, a, C\rangle$ be a ranking problem such that $a_i = a_j$. We need to show that $\phi(R) = (c_{ij}/(c_{ij} + c_{ji}), c_{ji}/(c_{ij} + c_{ji})) \equiv \phi_I(R)$. By invariance to citation intensity, we can assume that the entries of each column of C add up to one. By weak homogeneity, there is a positive constant α such that:

$$\phi_i(R)/\phi_j(R) = \alpha \frac{c_{ij}}{c_{ji}}. \tag{3}$$

We will show that $\alpha = 1$. To see this, consider the auxiliary ranking problem $R' = \langle \{i,j\}, a, C' \rangle$ with

$$C' = \begin{pmatrix} c_{jj} & c_{ji} \\ c_{ji} & c_{jj} \end{pmatrix}.$$

By weak homogeneity, $\phi_i(R')/\phi_j(R') = \alpha$. On the other hand, letting $P = \binom{0\,1}{1\,0}$, by anonymity:

$$\begin{aligned} P\phi(R') &= \phi(\{i,j\}, Pa, PC'P^T) \\ &= \phi(\{i,j\}, a, C') \\ &= \phi(R'), \end{aligned}$$

where the second equality follows from the particular choice of the matrix C'. Therefore, we must have that $\phi(R') = (1/2, 1/2)$ and, consequently, $\alpha = 1$. As a result, it follows from (3) that

$$\phi_i(R)/\phi_j(R) = \frac{c_{ij}}{c_{ji}}$$

which, together with $\phi_i(R) + \phi_j(R) = 1$, implies the desired result. □

The next Lemma allows us to determine the method for all isoarticle problems.

Lemma 2 Let $\phi : \mathcal{R} \to \Delta$ be a ranking method that satisfies weak homogeneity, invariance to citation intensity and weak consistency, and that coincides with the Invariant method for all two-journal, isoarticle problems. Then, ϕ coincides with the Invariant method for all isoarticle problems.

Proof : The proof is by induction on the number of journals. Assume that ϕ coincides with the Invariant method for all n-journal problems in which all journals have the same number of articles, and let $R = \langle J, a, C\rangle$ be an $n+1$ journal problem in that class. Since both ϕ and ϕ_I satisfy invariance to citation intensity, we can assume without loss of generality that the entries of each column of C add up to one. By the induction hypothesis, for all $k \in J$ we have $\phi(R^k) = \phi_I(R^k)$. But then, by weak consistency of both ϕ and ϕ_I, for all $k \in J$

$$\frac{\phi_i(R)}{\phi_j(R)} = \frac{\phi_i(R^k)}{\phi_j(R^k)} = \frac{(\phi_I)_i(R^k)}{(\phi_I)_j(R^k)} = \frac{(\phi_I)_i(R)}{(\phi_I)_j(R)} \qquad \text{for all } i, j \in J.$$

This implies that $\phi(R) = \phi_I(R)$. □

In order to complete the proof, we need to show that if ϕ satisfies all the axioms and coincides with the Invariant method for all isoarticle ranking problems, then ϕ is in fact the Invariant method.

Let $R = \langle J, (a_j)_{j\in J}, (c_{ij})_{(i,j)\in J\times J}\rangle$ be a ranking problem and let $A = \mathrm{diag}(a_j)_{j\in J}$. Let ϕ be a ranking method that satisfies all the foregoing axioms. We need to show that $\phi(R)$ solves the equation $Av = CD^{-1}Av$. Define $v^* \in \Delta_J$ to be the only solution to $CD^{-1}v = v$. That is, v^* would be the vector of relative valuations awarded by the Invariant method if all the journals in J had the same number of articles. We need to show that $\phi(R) = \frac{A^{-1}v^*}{\|A^{-1}v^*\|}$. Let G be the greatest common divisor of $(a_j)_{j\in J}$ and let $T_j = a_j/G$. We will split each journal $j \in J$ into T_j identical journals. The set of journals will be $J' = \{(j, t_j) : j \in J, t_j \in T_j\}$. The number of articles of journal (j, t_j), for $j \in J$, $t_j \in T_j$, is given by $a_{(j,t_j)} = a_j/T_j = G$. The new matrix of citations is $C' = (c'_{(i,t_i)(j,t_j)})$ where $c'_{(i,t_i)(j,t_j)} = c_{ij}/(T_iT_j)$. Summarizing, $R' = \langle J', (a_{(j,t_j)})_{j\in J, t_j\in T_j}, (c'_{(i,t_i)(j,t_j)})_{((i,t_i),(j,t_j))\in J'\times J'}\rangle$.

Since R' is a ranking problem where all journals have the same number of articles, we know by Lemma 2 that $\phi(R')$ is the solution to $C'D'^{-1}v = v$, where $C'D'^{-1}$ is the normalization of C'. Denote this unique solution by $\bar{v}$. Note that $\bar{v}$ is a $|J'|$-dimensional vector. However, by anonymity of f, we know that $\bar{v}_{(i,t_i)} = \bar{v}_{(i,s_i)}$ for all $i \in J$ and for all $t_i, s_i \in T_i$. That is, an article published in sub-journal (i, t_i) has the same value as an article published in the sub-journal (i, s_i). Denote this common value by $\bar{v}_i$, $i \in J$. We shall show that, for all $i \in J$, $T_i\bar{v}_i = \sum_{j\in J} T_j\bar{v}_j c_{ij}/c_j$. To see this, note that

$$\begin{aligned}
T_i\bar{v}_i = T_i\bar{v}_{(i,t_i)} &= T_i \sum_{j\in J}\sum_{t_j\in T_j} \bar{v}_{(j,t_j)} c'_{(i,t_i)(j,t_j)}/c'_{(j,t_j)} \\
&= T_i \sum_{j\in J}\sum_{t_j\in T_j} \bar{v}_j \frac{c_{ij}}{T_i T_j} T_j/c_j \\
&= \sum_{j\in J}\sum_{t_j\in T_j} \bar{v}_j c_{ij}/c_j \\
&= \sum_{j\in J} \bar{v}_j T_j c_{ji}/c_j.
\end{aligned}$$

Therefore, $\bar{v}_i T_i = v_i^*$. Dividing both sides by a_i we get

$$\frac{\bar{v}_i}{G} = \frac{v_i^*}{a_i} \qquad \text{for all } i \in J.$$

This means that the vectors $A^{-1}v^*$ and $(\bar{v}_i)_{i\in J}$ are proportional. But by the invariance of f to splitting of journals, we know that the vectors $\phi(R)$ and $(\bar{v}_i)_{i\in J}$ are proportional too, which implies that $A^{-1}v^*$ and $\phi(R)$ are proportional. Since $\|\phi(R)\| = 1$, we must have $\phi(R) = \frac{A^{-1}v^*}{\|A^{-1}v^*\|}$. □

2.2 Independence of the Axioms

This subsection shows that the five axioms used in the above characterization are logically independent.

In order to see that the weak homogeneity axiom is not implied by the other four, consider the Egalitarian method $\phi_E : \mathcal{R} \to \Delta$ defined by $\phi_E(J, a, C) = (1/|J|, \ldots, 1/|J|)^T$. It is easy to check that ϕ_E satisfies anonymity, invariance to citation intensity, consistency and invariance to splitting of journals, but it does not satisfy weak homogeneity.

We will now build a ranking method that satisfies all axioms except for weak consistency. Let $R = \langle J, a, C\rangle$ be a ranking problem. We say that journal $i \in J$ is *similar* to journal $j \in J$ if, $c_{i,k}/c_{i,s} = c_{j,k}/c_{j,s}$ and $c_{k,i}/c_{s,i} = c_{k,j}/c_{s,j}$ for all $k, s \in J$. The similarity relation is an equivalence relation. Therefore it partitions the set of journals into equivalence classes. Denote by $[i]$ the equivalence class that contains $i \in J$ and consider a ranking problem $R^* = \langle J^*, a^*, C^* \rangle$ where J^* is the set of equivalence classes of J induced by the above similarity relation,[10] $a^*_{[i]} = \sum_{t_i \in [i]} a_{t_i}$, for $[a_i] \in J^*$ and $c_{[i][j]} = \sum_{t_i \in [i]} \sum_{t_j \in [j]} c_{t_i t_j}$. That is, the ranking problem R^2 is obtained from R by merging all the journals that are similar to each other. Repeat the process starting from R^2 until the limiting ranking problem R^∞ has no two journals that are similar to each other. Consider the ranking method $\phi_{SC} : \mathcal{R} \to \Delta$ defined as follows. Given a ranking problem $R = \langle J, a, C \rangle$ let $\tilde{R} = \langle J, a, CD^{-1} \rangle$ be the associated normalized problem and let $\tilde{R}^\infty$ be the problem that results from the above merging of similar journals, applied to the normalized problem. The method ϕ_{SC} assigns to each journal $i \in J$ a value $v_i = \lambda v^*_{[i]}$, where $v^*_{[i]}$ is the value awarded to the equivalence class that contains $i \in J$ by the Modified Counting method when applied to the problem $\tilde{R}^\infty$ and λ is chosen so that $(v_i)_{i \in J} \in \Delta_J$. This method satisfies invariance to citation intensity since it applies the Modified Counting method to the normalized problem. It can be checked that it satisfies weak homogeneity and anonymity. It also satisfies invariance to splitting of journals, because by splitting a journal one gets similar subjournals. This method, however, does not satisfy weak consistency.

To see that anonymity is not implied by the other axioms, let $\sigma : \mathcal{J} \to I\!N$ be a non-constant function such that if (i, t_i) is a replica of $i \in \mathcal{J}$ then $\sigma(i) = \sigma(i, t_i)$. For each J let $H_J = \text{diag}(\sigma(j))_{j \in J}$. The function ϕ_σ assigns to each ranking problem $R = \langle J, a, C \rangle$, the unique solution $v \in \Delta_J$ to $A^{-1}(CD^{-1})AH_J v = H_J v$. By its own definition, the method ϕ_σ satisfies invariance to citation intensity. It can be checked that it also satisfies weak homogeneity: for all relevant problems $R = \langle J, a, C \rangle$,

$$(\phi_\sigma)_i(R)/(\phi_\sigma)_j(R) = \alpha c_{ij}/c_{ji},$$

where $\alpha = \sigma_i/\sigma_j$. The proof that this method satisfies consistency is analogous to the proof that the Invariant method satisfies consistency. This method also satisfies invariance to splitting of journals. The method ϕ_σ does not satisfy anonymity.

To see that the invariance to citation intensity axiom is not implied by the others, consider the Liebowitz-Palmer method: it satisfies anonymity, homogeneity, consistency and invariance to splitting of journals, but does not satisfy invariance to citation intensity. To see that it satisfies anonymity, let P be a permutation matrix and assume that v^* solves

[10] To be accurate, $J^* \subseteq J$ contains exactly one element of each equivalence class.

$A^{-1}Cv = \lambda v$, where $\lambda = \|A^{-1}Cv^*\|$ is the maximal eigenvalue of C. We need to show that Pv^* solves $(PA^{-1}P^T)(PCP^T)v = \lambda v$. But if $\lambda v^* = A^{-1}Cv^*$, then

$$\begin{aligned} \lambda Pv^* &= PA^{-1}Cv^* \\ &= (PA^{-1}P^T)(PCP^T)Pv^*, \end{aligned}$$

which means that Pv^* is a maximal eigenvector of $(P^TA^{-1}P)(PCP^T)$. The fact that this method satisfies homogeneity follows from the fact that for any matrix of the form

$$A^{-1}C = \begin{pmatrix} \kappa/\lambda_1 & 0 \\ 0 & \kappa/\lambda_2 \end{pmatrix} \begin{pmatrix} \lambda_1 - c_{21} & c_{12} \\ c_{21} & \lambda_2 - c_{12} \end{pmatrix}$$

we have $A^{-1}C(c_{12}, c_{21})^T = \kappa(c_{12}, c_{21})^T$.

To see that the Liebowitz-Palmer method satisfies consistency, let $(v_i^*)_{i\in J} = \phi_{LP}(R)$. That is $(v_i^*)_{i\in J} \in \Delta_J$ solves $v = \lambda A^{-1}Cv$, for some $\lambda \neq 0$. Since $DA^{-1} = \kappa I$, we have that

$$Dv^* = \lambda\kappa Cv^*.$$

The rest of the proof is analogous to the proof that the Invariant method satisfies consistency. The proof that this method satisfies invariance to splitting of journals is similar to the analogous proof for the Invariant method, and is left to the reader.

Lastly, consider the ranking method $\phi : \mathcal{R} \to \Delta$ defined by $\phi(J, a, C)$ is the only eigenvector in Δ_J of the matrix $(A^2)^{-1}CD^{-1}A^2$. This method satisfies anonymity, invariance to citation intensity, weak homogeneity and weak consistency. It does not satisfy invariance to splitting of journals.

3 Illustrations

We have examined the problem of ranking journals, web sites, patents and other entities. In this section we offer two applications of the Invariant method in the context of academic journals. We would like to stress that the only purpose of these applications is to illustrate the use of these methods and to compare them with the results obtained with other methods. Therefore, any similarities with the "true" rankings, that could be obtained only by considering the complete set of relevant journals over the relevant period, is pure coincidence.

We are particularly interested in comparing the per-manuscript and overall Invariant methods to other methods, as well as in calculating the implied impact of individual rather than representative papers. (The overall impact of a journal is simply its per manuscript impact multiplied by the number of its articles.) These comparisons could be illustrated using arbitrarily simulated data. Instead, we illustrate them using real disaggregated data that come from the citations given by each paper published in the year 2000 to every paper published during the period 1993–1999. Given the high cost involved in looking up and counting citations by hand from a large number of journals, we just examine a small set of journals.[11] The journals we examine are the following five general interest journals: *Review of Economic Studies, American Economic Review, Econometrica, Journal of Political Economy* and *Quarterly Journal of Economics*.[12] For purposes of comparison, in addition to the per-manuscript Invariant methods, we also calculate the numerical rankings that result from other commonly used methods. We will see that in this data set, the differences can be substantial. Table 1 first shows the aggregate data.

Table 1

Number of citations for five journals, 2000

Cited Journal	# of articles	Citing Journal				
		REStud	AER	Ecmta	JPE	QJE
Review of Economic Studies	252	21	24	15	13	10
American Economic Review	577	8	122	10	24	38
Econometrica	369	48	27	57	17	11
Journal of Political Economy	355	18	43	7	31	20
Quarterly Journal of Economics	294	13	73	10	24	53
Total		108	289	99	109	132

The following table shows the measure of overall impact for these journals according to the Invariant and the overall method introduced by Liebowitz and Palmer (1984).[13]

[11]The required data per article are not available in Journal Citation Reports or, to the best of our knowledge, in any other electronic database.

[12]The data we will use in what follows for the *American Economic Review* excludes the "Papers and Proceedings" issue published in May.

[13]This overall method is different from the (per-manuscript) Liebowitz-Palmer method discussed in the

Table 2
Overall Impact of Journal Rankings Based on Citations in 2000

Invariant method		Overall Liebowitz-Palmer method	
1. *Econometrica*	100.00	1. *American Economic Review*	100.00
2. *American Economic Review*	84.39	2. *Quarterly Journal of Economics*	79.22
3. *Quarterly Journal of Economics*	81.33	3. *Econometrica*	51.28
4. *Journal of Political Economy*	54.72	4. *Journal of Political Economy*	48.42
5. *Review of Economic Studies*	43.26	5. *Review of Economic Studies*	30.55

Note the substantial difference between the two numerical rankings. While, according to the Invariant method, Econometrica has substantially more impact than the AER and the QJE, according to the LP method the overall impact of Econometrica is barely half that of the AER and close to two thirds that of the QJE. This difference can be traced back to the failure of the LP method to satisfy invariance to citation intensity. Table 1 shows that the AER and QJE together have more cited references (a longer list of references) than the other three journals. As a result, since the LP method gives more weight to the opinions of journals with greater citation intensity, this method yields a ranking more biased toward the opinions of the AER and QJE. Note that these two journals give more weight to AER and QJE than to the other three.

Table 3 shows the impact per manuscript of each of the five journals. Note again the great difference in the measurement of impact. While QJE and Econometrica have similar impact according to the Invariant method, according to the Liebowitz-Palmer method a paper in Econometrica has 63% of the impact of a paper published in the QJE. Again, the differences in the numerical rankings are due to the failure of the Liebowitz-Palmer method to satisfy the invariance axiom.

previous sections. This method was proposed in Liebowitz and Palmer (1984) as is defined as the maximal eigenvector of the matrix of citations C.

Table 3
Per-Manuscript Impact of Journal Rankings Based on citations on 2000

Invariant method		Liebowitz-Palmer method	
1. *Quarterly Journal of Economics*	100.00	1. *Quarterly Journal of Economics*	100 .00
2. *Econometrica*	97.96	2. *Econometrica*	63.60
3. *Review of Economic Studies*	62.05	3. *American Economic Review*	55.06
4. *Journal of Political Economy*	55.72	4. *Journal of Political Economy*	51.00
5. *American Economic Review*	52.87	5. *Review of Economic Studies*	49.05

Lastly, a number of questions of interest require an analysis at the level of individual rather than representative manuscripts. For instance, in addition to the various issues mentioned in footnote 2, there is considerable discussion of the possible role of editors in steering disciplines, pushing or suppressing various lines of research, favoring friends and personal associates, and others. It also seems clear that articles may vary individually in quality and influence, and that not all articles appearing in the same journal, even in the same issue, are likely to be equally influential. A journal might, for example, provide the same impact by publishing all of its papers of similar quality, while another journal may attain the same overall impact by publishing just a few articles of exceptional quality and many of low quality.[14]

The Invariant method can also be used to calculate the impact of individual articles.

The average value of an article published in journal i is, according to the Invariant method, $v_i = \sum_j \frac{c_{ij}/a_i}{c_j/a_j} v_j$.

Therefore, the value of a particular paper p_i published in journal i is given by $v_{p_i} = \sum_j \frac{c_{p_i j}}{c_j/a_j} v_j$. Namely, the impact-adjusted sum of the citations of paper p_i.

Note that the average impact of a paper in journal i may be greater than that of a paper in journal j, while a citation from journal j may be more valuable than a citation from journal i. Formally, we may have $v_i > v_j$, while at the same time, $\frac{1}{c_j/a_j} > \frac{1}{c_i/a_i}$.

As an example, the following are the top five papers in our sample according to the Invariant method:

[14] It would then be relevant to examine differences across journals in the degree of inequality in the distribution of influence of the articles in the journal, using inequality criteria base on normalized stochastic dominance. Any difference may reflect, at least in part, differences in risk-taking behavior or other aspects of the tastes of the editors.

Table 4
Top five impact-adjusted cited articles, 2000

Cited Article	Impact
Benabou (1996)	3.02
Milgrom and Shannon (1994)	3.01
Ellison (1993)	2.90
Juhn, Murphy and Pierce (1993)	2.90
Autor, Katz and Krueger (1998)	2.83

Note that although, on average, a paper in the Quarterly Journal of Economics has more impact than a paper in the American Economic Review, the paper with most impact in our sample appeared in the AER. Also note that although a paper published in the QJE has more impact that one published in Econometrica, a citation from Econometrica is more valuable than a citation from QJE. This is so because the average paper in QJE has a longer reference list, thus awarding each of its cited references a small proportion of its value.[15]

4 Concluding Remarks

We have examined the problem of measuring influence using the information contained in the communication network between different entities. The same problem arises in the analysis of the basing of judicial decisions on previous decisions, the analysis of the dynamics of innovation where patent records contain citation references to previous patents and discoveries, in the World Wide Web where search engines rank web pages according to their importance, and in many other areas. The information provided by these communications is important. It allows us to make rigorous quantitative analyses of elusive but important phenomena, such as reputation, influence, the diffusion of scientific knowledge, the quality of journals, the productivity of scholars, changes in the publication process and

[15]The references for these papers are: R. Benabou (1996), "Heterogeneity, Stratification, and Growth: Macroeconomic Implications of Community Structure and School Finance," AER, pp. 584-609; P. Milgrom and C. Shannon (1994), "Monotone Comparative Statics," Econometrica, pp. 157-180; G. Ellison (1993), "Learning, Local Interaction, and Coordination," Econometrica, pp. 1047-1071; C. Juhn, K. M. Murphy, and B. Pierce (1993), "Wage Inequality and the Rise in Returns to Skill," JPE, pp. 410-442; D. H. Autor, L. F. Katz, and A. B. Krueger (1998), "Computing Inequality: Have Computers Changed the Labor Market?," QJE, pp. 1169-1213.

in the returns and incentives for publication, several aspects of the labor market for scientists, personnel decisions in academic administration, and many others. Yet, the actual use of this information has been conducted without any use of theoretical methodologies that could help us understand the content in these communications. In particular, simple citation counts and a number of arbitrary methods have been widely employed in the measurement of influence and in applications.

The result of these arbitrary practices is that the information content of the network communications that play a paramount role in the exchange, dissemination, and certification of knowledge and information is little understood. This, in turn, has important implications for all the empirical applications mentioned above.

In this paper we apply the axiomatic methodology to bear on the problem of measuring influence based on communication data, and characterize different measurement methods according to the properties that they satisfy and those that they do not satisfy. We obtain that a unique ranking method can be characterized by certain basic properties.

Lastly, we would like to stress that the analysis of communication data cannot be a substitute for critical reading and expert judging, but rather a complement. The communication between scholarly journals, legal opinions, patents, web pages, and other publications contains valuable information that can be used to address several empirical questions of interest. We have argued that we should ask why a given methodology to measure influence is reasonable and informative.

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