Measurement of Welfare Change: A Review

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MEASUREMENT OF WELFARE CHANGE: A REVIEW

This paper reviews theory for measuring welfare changes for a single consumer.[1] The first section deals with the notion of consumer surplus, whereas the second section focuses on the concepts of compensating variation and equivalent variation. These three measures are often used to evaluate the welfare implications of a policy change.

Consumer Surplus

Assumptions about consumer behavior are introduced into demand theory through the specification of a utility function. The utility function measures the level of satisfaction an individual experiences as a result of consuming a particular basket of commodities. The utility function is denoted as

$$u = u(q),$$

where $q = (q_i)$ is taken to be an n-element vector whose elements are levels of commodities consumed per unit of time.

The utility function is maximized subject to the following budget constraint, which specifies that available income is exactly spent:

$$p'q = m \ ,$$

where $p = (p_i)$ is the n-element column vector of prices and m is consumer income.

Maximizing the utility function, (1), subject to the budget constraint (2) is carried out by the Lagrangian method such that

$$L(q, \lambda) = u(q) \lambda(p'qm),$$

where λ is the Lagrangian multiplier interpreted as the

marginal utility of income. Differentiating the Lagrangian equation, (3), with respect to each of the arguments, q_i and λ, yields the first-order conditions

$$u_q \, \lambda p = 0, \qquad p'qm = 0,$$

where u_q is the vector of derivatives of the utility function with respect to the quantities, q_i, $i = 1, ..., n$.

The system of equations obtained from the first-order conditions in (4) provides $n + 1$ equations in $2n + 2$ variables: the n prices, the n quantities, λ, and income. By applying the implicit function theorem, (4) can be solved uniquely for $q_1, ..., q_n$ and λ in terms of prices and income. The resulting expressions are:

$$q_i = q_i(p_1, ..., p_n, m) \quad i = 1, ..., n$$

$$\lambda = \lambda(p_1, ..., p_n, m).$$

Substituting the demand equations (5a) into the utility function expressed in (1) gives the indirect utility function

$$u = u[q(p,m)] = v(p,m).$$

Equation (6) specifies the maximum attainable utility level for a given set of prices and a particular income. The advantage of considering the consumer demand problem in this context is the ease with which the demand equations can be derived. In particular, the Marshallian (uncompensated) demand function for the ith commodity is obtained by differentiating the indirect utility function, (6), with respect to prices and income and applying Roy's identity:

$$[\partial v(p,m)/\partial p_i]/[\partial v(p,m)/\partial m] = q_i(p,m) \quad i = 1, ..., n$$

That is, the Marshallian demand function gives the quantity demanded as a function of prices, holding income constant while changing the level of utility.

To obtain a monetary measure of utility change, the effect of small changes in prices and income is derived. Small changes in prices and income can be obtained by total differentiation of the indirect utility function, (6), and the application of Roy's identity such that

$$dv = \Sigma(\partial v/\partial p_i)\, dp_i + (\partial v/\partial m)dm$$
$$= \lambda[dm\Sigma\, q_i(p,m)\, dp_i].$$

Assuming that $\lambda = \partial v/\partial m > 0$ is constant and dividing both sides of (8) by λ, we obtain a monetary measure of utility change, i.e., a measure of welfare change such that

$$dw = dv/\lambda = dm[\sum_i q_i(p,m)\, dp_i].$$

For discrete changes in prices and income, the measure of welfare change, (9), can be written as

$$\Delta w = \Delta m \ [\sum_i q_i(p,m)\, \Delta p_i]$$
$$= \Delta m \ [\sum_i \int_0^1 q_i(p,m)\, dp_i]$$

Marshall called this measure of welfare change the consumer surplus (CS). In Figure 1, the CS is defined as the area under the demand curve and above the price line; i.e., the area of triangle p_1EC.

Two concerns about the uniqueness of the concept of the CS have surfaced. First, the value of the line integral (second term in [10]) depends upon the order in which price and income change; i.e., the order in which the integration is performed. This implies that the line integral is not path independent. Consequently, the CS concept is not unique. Path independency of the line integral (and hence the uniqueness of the CS) holds only if cross-price derivatives $\partial q_i/\partial p_j = \partial q_j/\partial p_i$ of the
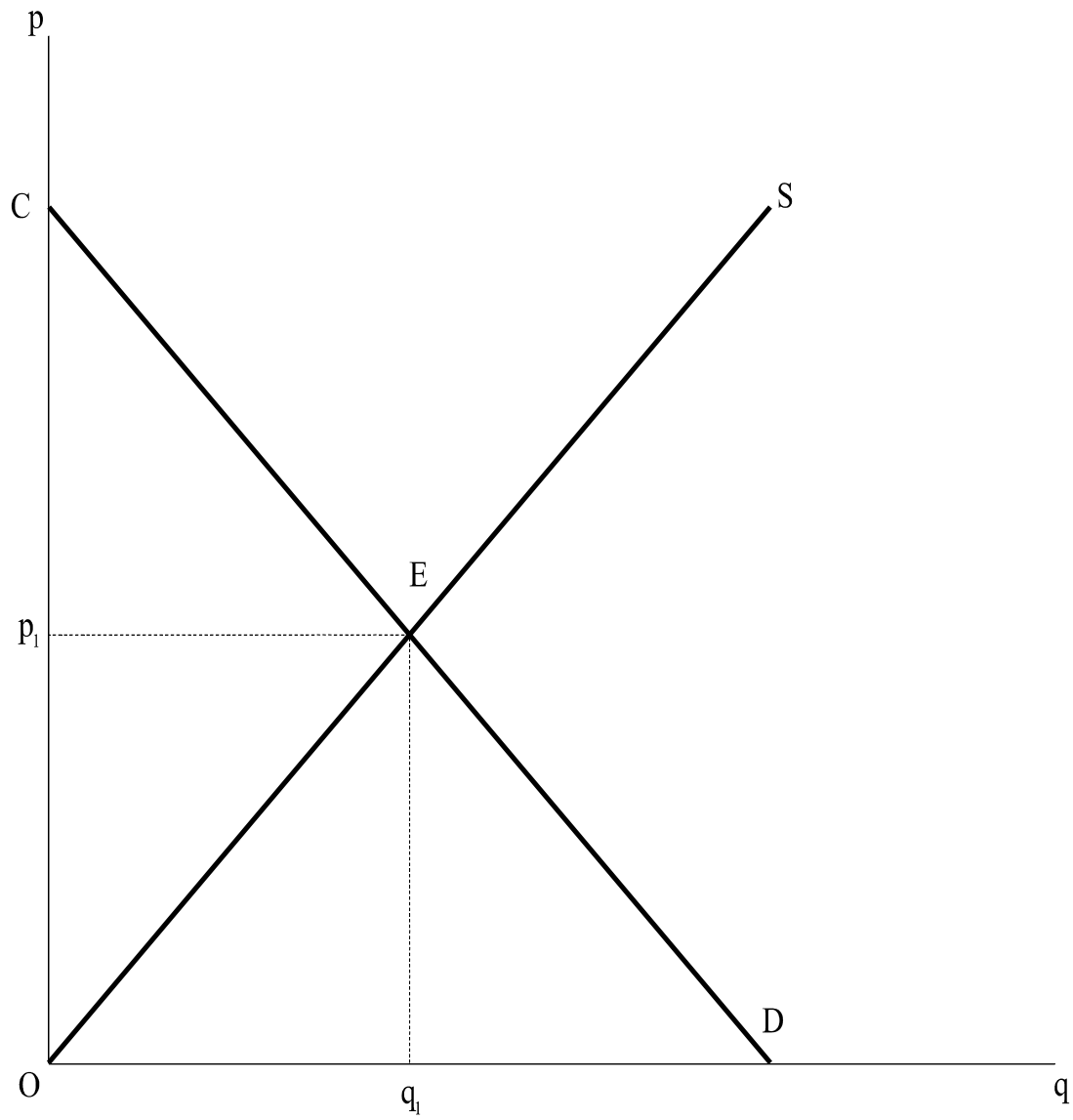
Figure 1. Consumer surplus

demand functions are symmetrical; i.e., the demand curves are compensated. Second, the marginal utility of income, λ, can be constant with respect to all prices but not income, or with respect to income and the first n! 1 prices. This implies that a unique measure of CS can be obtained under the following conditions.

1. If one price remains constant (the numéraire, the price of q_n) and income and the remaining prices change, then (10) reduces to

$$CS = \Delta m \ [\sum_{i=1}^{n \ 1} \int_0^1 q_i(p,m) \, dp_i]$$

Figures 2a and 2b show that, as income is changed from m_0 to m_1 at price p_1, the quantity demanded remains at q_1. Similarly, as income is changed from m_1 to m_2 at price p_2, the quantity demanded remains at q_2. The area of consumer surplus is given by p_1p_2ba (Figure 2b). Uniqueness under this condition is therefore obtained if the income elasticities associated with prices that change are zero. This condition implies that the indifference curves are vertically parallel.

(2.) If income remains constant and all prices change, then (10) reduces to

$$CS = \sum_{i=1}^{n} \int_0^1 q_i(p,m) \, dp_i .$$

This case is illustrated in Figures 3a and 3b. Specifically, Figures 3a and 3b assume that the price of q_1 falls from $p1$, to $p2$, and that all goods other than q_1 are aggregated into q_n. It shows that points 1 and 2 (Figure 3a) correspond to points $(p1, q1)$ and $(p2, q2)$ on the Marshallian demand curve $q_1(p_1)$ (Figure 3b). The area of consumer surplus is denoted by $p1p2ba$. A unique measure of welfare change is obtained under this condition if and only if preferences are homothetic. Homothetic preferences produce demand curves with unitary income elasticities.
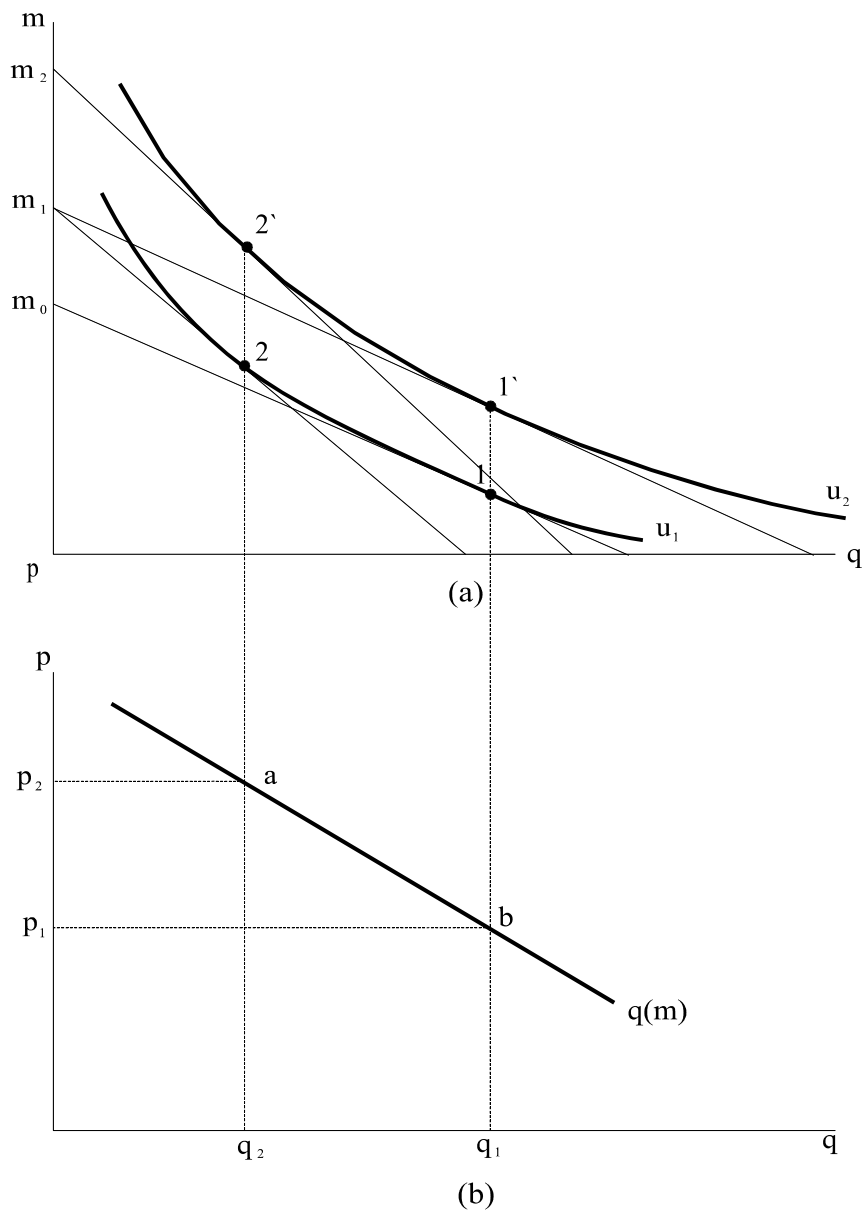
(a)

Axis labels for graph (a): m (vertical), q (horizontal), p
m_2, m_1, m_0
Points: 2`, 2, 1`, 1
Curves: u_1, u_2

(b)

Axis labels for graph (b): p (vertical), q (horizontal)
p_2, p_1
Points: a, b
q_2, q_1
Curve: q(m)

Figure 2. Consumer surplus: One price remains constant and income and the remaining prices change

q_n

m

1

2

u_2

u_1

q_1

(a)

p_1

p_1^1

a

p_1^2

b

$q_1 (p_1)$
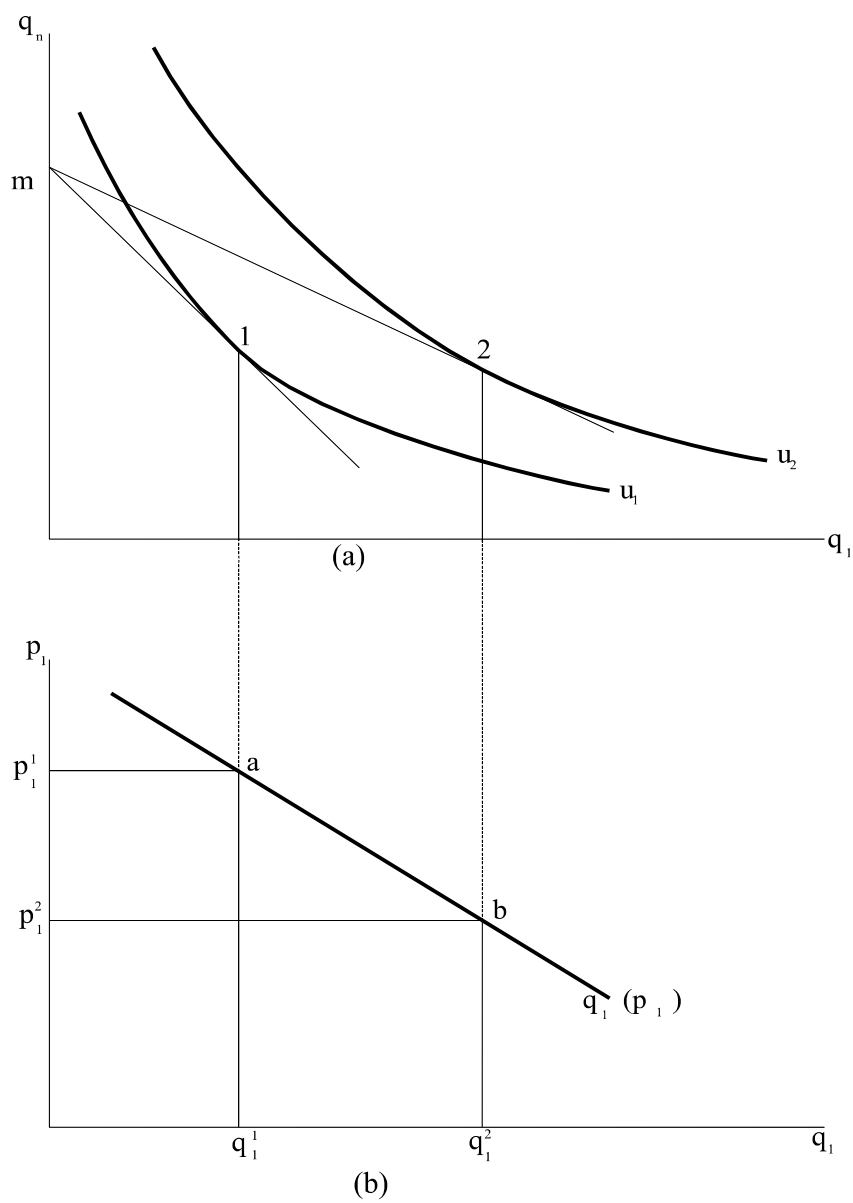
q_1^1

q_1^2

q_1

(b)

Figure 3. Consumer surplus: One price changes and income and the remaining prices remain constant

The previous discussion implies that consumer surplus is considered as a unique measure of utility change when the marginal utility of income is constant. That is, constancy of marginal utility of income guarantees path independency of the line integral, whereas path independency of the line integral does not guarantee constancy of marginal utility of income. Because of these restrictive assumptions concerning the constancy of the marginal utility of income and the question of path dependency of the line integral, alternative measures that do not suffer from these defects have been advanced, including the compensating and equivalent variations.

Willingness-to-Pay Measures

Before discussing the concepts of compensating and equivalent variations[2], it would be useful to introduce a utility function that is frequently used in demand analysis. This function is closely related to the indirect utility function and is known as the *expenditure (cost) function*. The expenditure function is obtained by inverting the indirect utility function, (6), and solving for m in terms of the level of utility u^o and a set of prices p.

More formally, the cost of attaining utility level u^o with prices p is

$$e(u,p) = \min_q p'q \text{ subject to } u(q) \geq u^o \ .$$

The expenditure function, (11), gives the minimum expenditure of achieving a utility level u^o at prices p.

Roy's identity was applied in the previous section to derive the Marshallian (uncompensated) demand functions from the indirect utility function. Similarly, the Shephard Lemma can be applied to yield the Hicksian (compensated) demand functions from the expenditure function. That is, the compensated demand function follows from differentiating expenditure function (11) with respect to p_i:

$$\frac{\partial e(u,p)}{\partial P_i} = q_i^*(u,p) \quad i = 1,...,n.$$

Thus, the Hicksian demand function gives the quantity demanded as a function of prices, holding utility constant while changing the level of income.

Compensating Variation

Compensating variation (CV) is defined as the amount of income that must be taken away from a consumer (positive or negative) after an economic change to restore him to the original welfare level. In other words, the CV is the income adjustment required to maintain the consumer at the utility level that occurred *before* price and income changes. In Figure 4, the consumer is initially at point 1 on the indifference curve u_1. If the price of a normal commodity, q_1, is lowered, the consumer moves to point 2 on the indifference curve u_2. The CV associated with the price and the income change is given by m_2 ! e_1. The CV is the income adjustment required to make the consumer indifferent between consuming the original basket at point 1 and facing the lower price but consuming the basket at point 2.

More formally, the CV for a change in prices and income from $(p1,m_1)$ to $(p2,m_2)$ can be written as

$$CV = m_2 \ e_1$$

$$= m_2 \ e(\ p_2 , u_1)$$

$$= m_2 m_1 + m_1 \ e(\ p_2 , u_1)$$

$$= \Delta m + e(\ p_1 , u_1) \ e(\ p_2 , u_1)$$

$$= \Delta m \ \int_{p_1}^{p_2} \sum_i [\ \partial e(p,uSUB1)/\partial \ p_i]\ dp_i$$

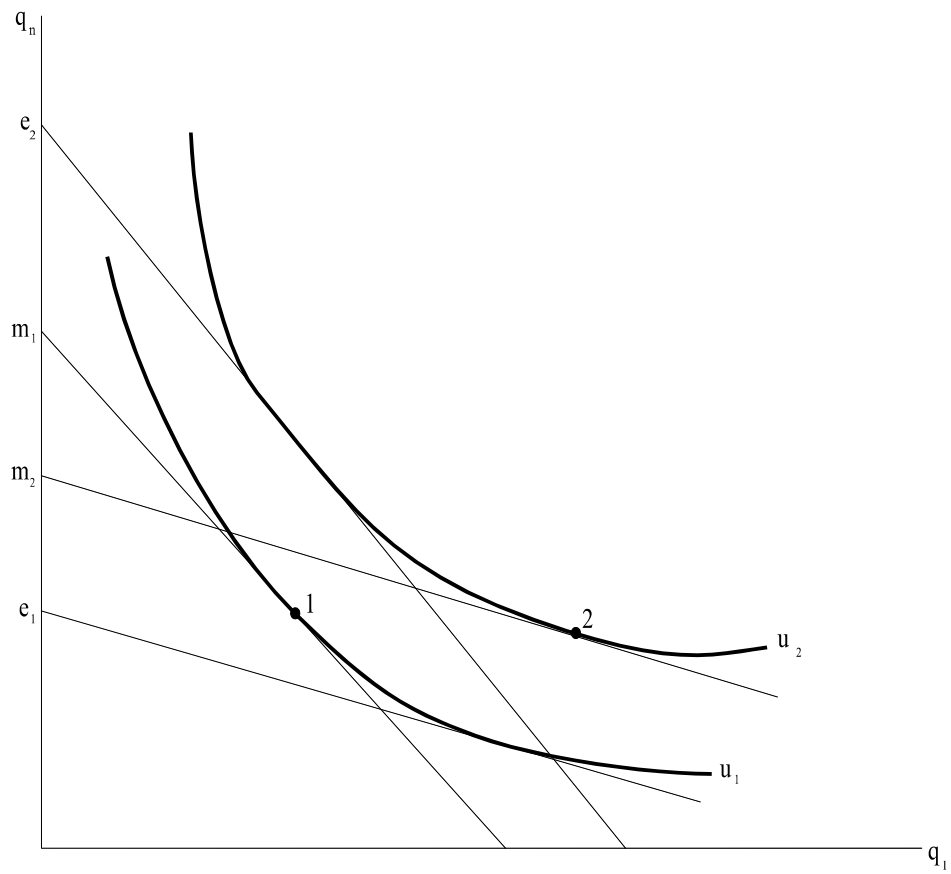$$= \Delta m \ \int_{p_1}^{p_2} \sum_i q_i (p,uSUBi)\ dp_i \ .$$

Figure 4. Compensating and equivalent variations

The question of path dependency does not arise here because the cross-price derivatives of the compensated demand functions are symmetrical.

If the price of a normal commodity, q_i, changes while all other prices and income remain constant, then CV in (13) reduces to

$$CV = \int_{p_i}^{p_i^2} q_i(p,u_1)dp \, SUBi \, .$$

This case is illustrated in Figures 5a and 5b for a decline in price of a normal commodity, q_1. The compensated demand and uncompensated demand curves for a price decline are given in Figure 5b. Also, Figure 5b shows that the CV for this particular case is equal to areas $(1 + 2)$, whereas the consumer surplus is equal to areas $(1 + 2 + 3)$.

Equivalent Variation

In contrast to the CV, the equivalent variation (EV) is defined as the amount of income that must be given to a consumer (positive or negative) in lieu of an economic change to make him as well off as with the change.

Figure 4 illustrates the EV for a move from point 1 to point 2. In contrast to the CV, the EV uses the level of utility *after* price and income changes as a basis. Thus, the EV for this situation is equal to the distance $e_2 \, ! \, m_1$ (see Figure 4). That is,

$$EV = e_2 \, m_1$$

$$= e(p_1,u_2) \, m_1$$

$$= e(p_1,u_2) \, m_2 + m_2 \, m_1$$

$$= e(p_1,u_2) \, e(p_2,u_2) + m_2 \, m_1$$

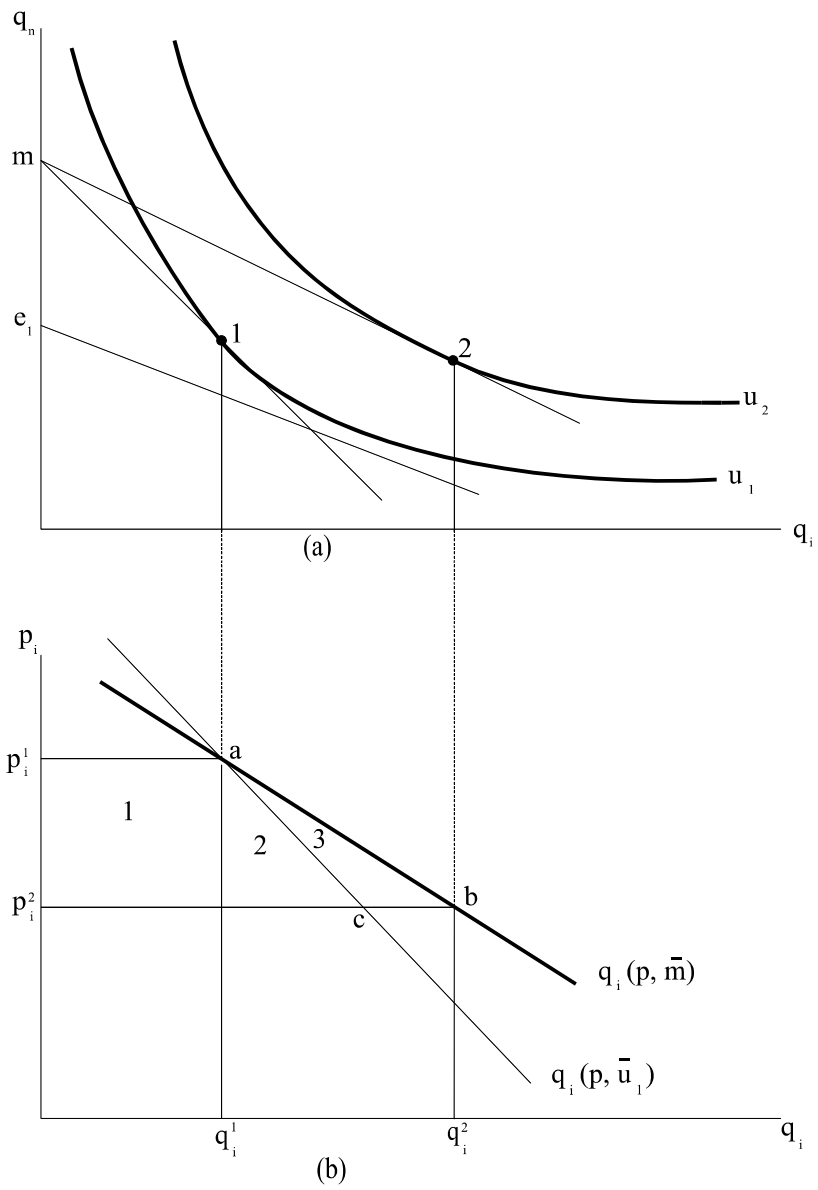$$= e(p_1,u_2) \, e(p_2,u_2) + \Delta m$$

Figure 5. Compensation variation: One price change while other prices and income remain constant

$$= \int_{p_1}^{p_2} \sum_i \left[\partial e(p,u_2)/\partial p_i \right] dp_i + \Delta m \ .$$

$$= \int_{p_1}^{p_2} \sum_i q_i \big(p,u_2\big) dp_i + \Delta m \ .$$

Again, as with the CV, the path dependency question does not arise because the cross-price derivatives of the compensated demand functions are symmetrical.

For a decrease in the price of a normal commodity, q_i, the EV in (14) is written as

$$EV = \int_{p_i^1}^{p_i^2} q_i(p,u_2) dp_i \ .$$

This situation is illustrated in Figures 6a and 6b. In Figure 6b, the EV is equal to area $(1 + 2 + 3)$, whereas the CS is equal to area $(1 + 2)$.

Summary

The relationships among the three concepts (CS, CV, EV) are shown in Figure 7. In particular, Figure 7 shows that for the case of a normal commodity, q_i, the uncompensated demand curve is flatter than the compensated demand curves. Specifically, Figure 7 indicates that if the price of a normal good, q_i, decreases from p_1, to p_2, then CV < CS < EV[3]. Of course, for a price increase, the relationship would be EV < CS < CV. For the case of zero income effects, the three measures coincide; i.e., CV = CS = EV.

q_n

e_2

m

1

2

u_2
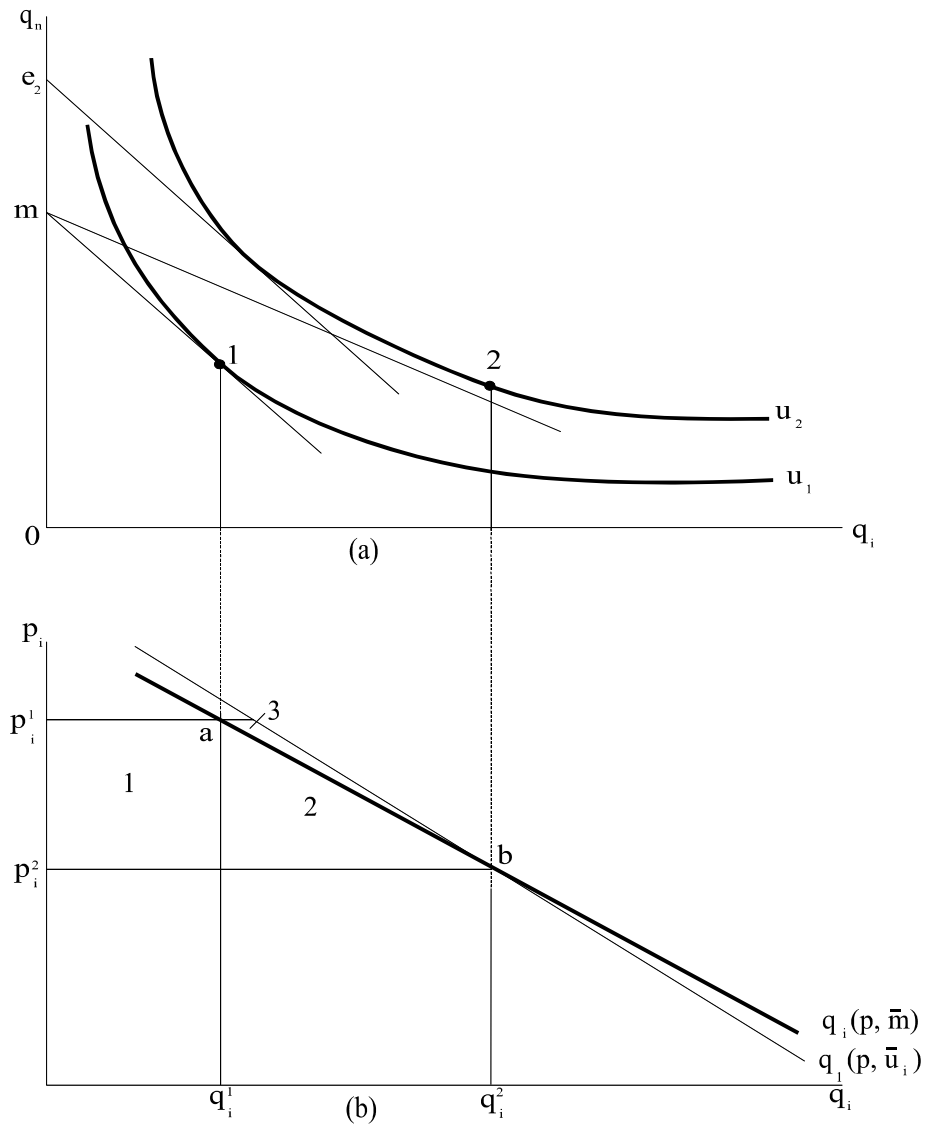
u_1

0

q_i

(a)

p_i

p_i^1

3

a

1

2

p_i^2

b

$q_i(p, \bar{m})$

$q_i(p, \bar{u}_i)$

q_i^1 q_i^2 q_i

(b)

Figure 6. Equivalent variation: One price changes, while other prices and income remain constant

p_i

p_i^2 .... a .... b

1

4

3

p_i^2 .... 2 ..... d c

$q_i(p_i, \bar{m})$

$q_i(p_i, \bar{u}_2)$

$q_i(p_i, \bar{u}_1)$
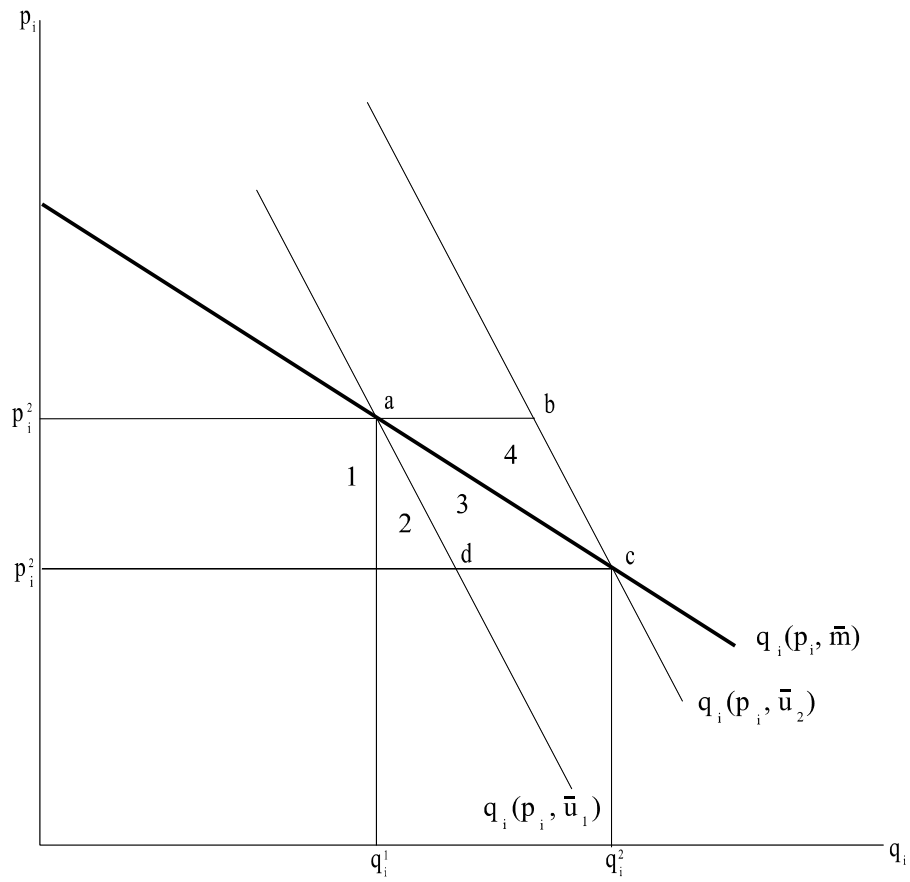
q_i^1 q_i^2 q_i

Figure 7. Relationships among consumer surplus, compensation variation and equivalent variation with a single price change

ENDNOTES

1. This review concentrates on consumer welfare measurement. For treatment of the concept of producer surplus/quasi rent, see Just, Hueth, and Schmitz (Chapters 4 and 7 and Appendix B).

2. For a discussion on the approximation of compensating and equivalent variations, based on consumer surplus, see articles by Hausman, Willig, Randall and Stoll, Hanemann, and Shogren et al.

3. The three concepts have the same sign as the direction of change in welfare. That is, for a welfare gain, the three measures are positive.

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