A Noncooperative Model of Collective Decision Making: A Multilateral Bargaining Approach

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SECTION 1. INTRODUCTION.

This paper proposes a noncooperative model of multilateral bargaining. The model can be viewed as an extension of the famous Stahl-Rubinstein bargaining game.[1] Two players take turns proposing a division of a "pie." After one player has proposed a division, the other can accept or reject the proposal. If the proposal is accepted, the game ends and the division is adopted; if it is rejected, the second player then makes a proposal, which the first player then accepts or rejects. And so on. An Stahl's formulation, the game continues for a finite number of rounds; in Rubinstein's extension, the number of rounds is infinite. We propose a generalization of this model to incorporate multiple players and multidimensional issue spaces. We consider a sequence of games with finite bargaining horizons, and study the limit points of the equilibrium outcomes as the horizon is extended without bound. A novel feature of our model is that the proposer is chosen randomly "by nature" in each round of bargaining, according to a prespecified vector of strictly positive "access probabilities."[2]

The present study departs from the Stahl-Rubinstein tradition in terms of the kinds of problems that are addressed. In this tradition, the object of negotiations is the division of a purely *private* good. In our paper, we focus instead on problems that relate to collective decision-making.[3] There is an enormous range of interesting and complex collective decision-making issues that, potentially, can be viewed from a multilateral bargaining perspective.[4] Consider, for example, the current debate among the formerly Soviet Republics over the fate of the Soviet Union, or the recent negotiations in Canada leading up to the Meech Lake Accord. Alternatively, imagine negotiations between regional interests within California over, say, the location of a new hydroelectric facility, or between members of an agricultural cooperative over the location of a new processing plant.

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[1] Stahl [1972, 1977] and Rubinstein [1981].

[2] Binmore [1987] also considers random selection as an alternative to the alternating-offer model. He discusses a two player game in which each is selected with probability one-half.

[3] It must be emphasized that there is nothing in the *formalism* of the model that requires us to restrict our attention to one particular class of problems. Rather, our model was developed with certain collective decision-making problems in mind, and, not coincidentally, provides more insights into such problems than others.

[4] For an actual application of the model, see Rausser-Simon [1991c], in which we use the framework developed in this paper as a basis for studying the process of privatization in Eastern Europe.

More abstractly, consider the issue of providing an indivisible *public* good. Suppose that the attributes of the alternative choices are summarized by points in the horizontal plane, over which the negotiators' preferences are ordered. Assume that the location decision will be determined by majority rule. This problem, known as the *spatial voting problem*, has spawned a vast research effort, which has been reported mainly in political science journals.[5] Certain interesting aspects of this problem simply do not arise in the traditional pie-division problem. In particular, as in many actual multilateral bargaining situations, the interests of the various parties are typically interrelated. What compromises will emerge as alliances are forged between parties whose interests are similar but not coincident? How effective will these alliances be in furthering the common interests of their members? What is the relationship between the "internal" alignment of interests within a given alliance and its "external" effectiveness as it negotiates with other alliances?

Our model is intended to address such issues. Indeed, it may be viewed as a stylized representation of the kind of unstructured "backroom" negotiations that might take place between members of the inner circle of a complex organization. Imagine, for example, that there is an important meeting scheduled for the plenary body of this organization (e.g., a parliamentary debate on a significant bill, a shareholders' meeting, etc.). Prior to this meeting, intense activity within the inner circle might be expected: coalitions would be formed, deals would be struck and compromises would be negotiated in informal, private, off-the-record meetings between influential members of the organization.

For example, imagine the informal negotiations between senior members of the White House staff over the selection of a nominee for a senior appointment (such as a Supreme Court judgeship). The following scenario might unfold: a number of different staff members, including, perhaps, the President himself, are concurrently lobbying each other, each attempting to build support for one particular candidate; somehow, one of the candidates is singled out from the others and, in a plenary meeting of the White House staff, attention is focussed exclusively on this candidate. If sufficient support has been generated for the candidate, then the White House will adopt him or her as its official

[5] See Fiorina-Plott [1978] for a seminal contribution and Harrison [1991b] for a recent survey.

nominee. Otherwise, the lobbying process will begin again, until agreement is finally reached.

The formalism of our model conforms rather closely to this informal negotiating process. One aspect of it, however, is difficult to describe analytically: how is one of the candidate singled out from the others? In our model, this problematic issue is "black boxed:" we simply assert that nature chooses between proposals in a random way. Presumably, however, there is some relationship between a staffer's relative political power within the organization and the likelihood that his or her proposal will be singled out for consideration. To formalize this idea, we assume that nature's random choice is governed by an exogenously specified vector of *access probabilities.* The access probability assigned to each participant is interpreted as a measure of his or her relative political power.

Outline of the Paper.

Our formal model is presented in Sections II and III. The content of these two sections is summarized in a heuristic way below. Our game consists of a finite number of *negotiating rounds.* The purpose of negotations is to select a *policy* from some set of possible alternatives. In odd-numbered rounds, each player chooses a *proposal,* which is a policy, paired with an *admissible coalition.* Between the odd- and even-numbered rounds, one of these proposals is selected by nature, according to the prespecified vector of access probabilities. In the even-numbered rounds, each member of the proposed coalition decides whether to accept or reject the proposed policy. The game ends as soon as all coalition members accept a policy. If one member rejects a policy, the players proceed to the next round. If the last round of the game is reached and the players still fail to agree, then the game ends and a *disagreement outcome* is implemented.

An important parameter of the model is the set of admissible coalitions. An admissible coalition is interpreted as a subset of the players that has the authority to impose a policy choice on the whole group. For example, in a *majority rule bargaining game,* a coalition is admissible if and only if it contains a majority of players. More generally, the set of admissible coalitions might have a variety of structures. In particular, we will sometimes impose the restriction that at least one player belongs to *every* admissible coalition. Any such player will be referred to as *essential.* For example, a *unanimity game* is a special case in which every player is essential.

We define an equilibrium concept which is a refinement of subgame perfection (Selten [1975]). For a bargaining game with a fixed number of bargaining rounds, an *equilibrium outcome* is a probability distribution over the policies that are implemented when players play equilibrium strategies. A *solution* to our bargaining model is a limit of equilibrium outcomes, as the number of negotiating rounds is increased without bound. Our main results concern the existence of a *deterministic* solution, which is a limit outcome assigning probability one to a single policy. A necessary condition for a policy to be a solution is that the policy belongs to the *core* of the underlying bargaining game. In this case, there exists no admissible coalition whose members all prefer some other policy. We identify two kinds of sufficient conditions for existence. If all players are risk averse, then every majority rule bargaining game with a one-dimensional space of policies has a deterministic solution. Alternatively, for general policy spaces, a deterministic solution exists if at least one player is essential. This alternative restriction is satisfied by an important class of games that has been widely analyzed: unanimity games. In such games, the only admissible coalition is the grand coalition, and each player is essential. Weak conditions guarantee that when a solution exists, it will be unique for generic specifications of players' preferences. An appealing feature of the model is that even for quite complex specifications of the bargaining problem, the equilibrium solution is relatively easy to compute numerically.

Section IV contains examples, designed to highlight strengths and weaknesses of the model and to motivate potential applications. Most of the examples are spatial models. The richest specifications are those in which the core of the underlying game is infinite. In these cases, some quite subtle comparative statics issues arise, such as the sensitivity of the model's solution to changes in the bargaining attributes of the players, to changes in the alignment of players' preferences, and to changes in the structure of the set of admissible coalitions. We investigate some of these relationships in an extremely simple, bipolar model of political conflict, and relate the bargaining performance of the left- and right-wing to various factors, such as the relative cohesiveness of the two factions. Next, we briefly explore the implications of introducing one or more essential players into the analysis. Finally, in a radical departure, we discuss multilateral bargaining in a very simple, four-person pure exchange economy.

Modelling Issues.

Because our sufficiency conditions appear severely restrictive, we begin with some remarks on the question of existence. First, we have overwhelming evidence based on exhaustive numerical simulations that deterministic solutions exist under significantly weaker conditions than the ones identified in the paper. Second, our simulations indicate that still weaker conditions will guarantee the existence of solutions that may not be deterministic.[6] Third, the fact that not all games have solutions need not necessarily be interpreted as a shortcoming of the model. Rather, it may be an indication that in certain environments, multilateral negotiation processes may be inherently unstable.

Quite apart from the existence question, there are certain multilateral bargaining problems about which not much of interest can be learned from the application of our framework. This is particularly true of the classical problem of dividing a purely private good, i.e., pie, among many players: if no player is essential, no solution exists; if exactly one player is essential, than this player receives the entire pie; if every player is essential then the pie is divided in the obvious way, i.e., proportional to players' access probabilities. It is because results such as these are neither surprising nor particularly interesting that we turn our attention to collective-decision making problems in which there is a nontrivial interaction between the interests of the various players.

A related topic concerns our sufficiency condition that some player be essential. In the abstract, this condition is quite restrictive and appears difficult to motivate (although, as we have noted, it is satisfied trivially in unanimity games). For example, it clearly conflicts with the formal institutional procedure of decision-making by majority rule. In spite of this, we maintain that in a wide variety of collective decision-making contexts, the condition is satisfied *de facto,* sometimes even when it is explicitly violated *de jure*. For example, recall our earlier scenario of informal White House negotiations. It is surely difficult to imagine that a candidate could emerge as the White House nominee for a major political appointment without at least the tacit approval of the President. In this context, the President would satisfy the criterion for an essential player. Similarly, if we were to

[6] As yet, we do not have analytical results that correspond to these observations.

model the current negotiations over the demise of the Soviet Union, it would not seem unreasonable to confer essential status on either Mr Gorbachev, Mr Yeltsin or both. More generally, whenever a group of negotiators has a clearly identified "leader," it will be natural to declare this player to be essential.[7] Moreover, a player might be granted essential status by virtue of her role in executing the decisions resulting from the negotiations. For example, consider a faculty meeting of a university department. Even when the Chairperson is only "one among equals," and has no special voting privileges, there are presumably certain certain kinds of policy decisions that will rarely be taken in the face of explicit opposition from the Chair, if for no other reason other than that the Chair alone will be responsible for implementing the policy.

We conclude this discussion section with some remarks on our treatment of the time horizon. Since Rubinstein [1981], it has become customary in bargaining models to assume that the time horizon is infinite. We depart from this trend, and assume that the bargaining horizon is finite but arbitrarily long. A pragmatic justification for this assumption is that the infinite-horizon version of our model has no predictive power: any outcome can be supported as an equilibrium.[8] More significantly, however, there are circumstances in which our finite-horizon approach seems called for on modelling grounds. These circumstances are especially likely to arise in the kinds of collective decision-making contexts that we have described. It is commonplace that in such contexts the presence of an impending deadline can provide a dramatic impetus to compromise: witness the frequency of last-minute resolutions of Congressional deadlocks, and of post-midnight compromises in wage negotiations when strikes are threatened for the following morning. In a finite horizon model, attention is inevitably drawn to these "eleventh hour" effects; conversely, in an infinite horizon model there is, of course, no endgame. Now in general, the profession is justifiably skeptical of results that rely heavily on long and intricate inductive chains. In our model, however, this skepticism may be mitigated somewhat by the fact that in many instances, the basic "shape" of the solution is clearly determined after only a few rounds of induction (often as few as three).[9] This fact may also reassure

[7] Conversely, in the absence of leadership, one might expect certain kinds of negotiations to become bogged down in vacillations; this might well be mirrored by the failure of a solution to exist in our model.

[8] The method of proof is discussed briefly in the discussion of "Related Literature" below.

[9] This point is demonstrated quite clearly when we discuss comparative statics in Section IV.

experimentalists, since there is abundant evidence that experimental subjects seem unable to backward induct much beyond three periods.[10]

Related Literature.

Until very recently, the topic of multilateral bargaining has received surprisingly little attention by noncooperative game theorists. The few papers which have been written focus almost exclusively on various versions of the alternating-offer model. Binmore [1985] considers several alternative extensions of Rubinstein's analysis to the problem of "three player and three cakes:" each pair of players exercises control over the division of a different cake, only one of which can be divided. In unpublished work, Shaked observed that in any infinite-horizon, alternating-offer, multilateral pure-division problem, if the consent of three players is required for agreement, and if they are not extremely impatient, then any division of the pie can be implemented by subgame perfect equilibrium strategies.[11] The proof follows easily from the following observation: suppose one player proposes an off-the-equilibrium-path division that gives her a positive share of the pie. If players are not too impatient, then at least one of the other two players can be induced to reject this division by the promise of the whole pie in the subgame that will follow if she does so.

An interesting variant of the alternating-offer model, called the "Proposal-Making Model," has been advanced by Selten [1981]. A player is selected by nature to make the first proposal. She proposes a utility vector, a coalition and a "responder." The responder either accepts or rejects. If she rejects, the responder then proposes a new utility vector, a new coalition and a new responder. If she accepts, the responder designates another member of the coalition as the next responder, and so on until all members of a coalition have agreed to some proposal. This model has been studied extensively in Chatterjee et al. [1987] and in a series of papers by Bennett and coauthors.[12]

Chae-Yang [1988, 1989] and Krishna-Serrano [1991] investigate the possibility that an individual player can unilaterally break off negotiations and exit, taking with her the share of the total pie that she

[10] See Binmore et. al. [1985], Neelin et. al. [1988] and Spiegel et. al. [1990]. See, however, Harrison-McCabe [1992] for a dissenting opinion, and Harrison [1991] for a survey.

[11] This result is discussed in Sutton [1986] and Osborne-Rubinstein [1990].

[12] Bennett [1991a, 1991b], Bennett and van Damme [1991] Bennett and Houba [1991].

has been able to negotiate. As Krishna-Serrano argue, a natural criterion that an equilibrium should satisfy in this context is a notion of "consistency" due to Lensberg [1988]. We will discuss this concept in some depth because it highlights the striking difference between our collective decision-making context and that of the classical pie-divison problem.[13] Loosely, a system of solutions to a family of multilateral bargaining problems is said to be *consistent* if the solution that players other than i obtain when i is present at the bargaining table is the same as the solution they obtain when i is absent and the total pie is depleted by the portion that i would receive if she were present. Essentially, this axiom formalizes the idea that the bargaining attributes of players are pairwise orthogonal. That is, there are no synergies between players: the relative bargaining strengths of players j and k cannot depend on whether or not player i is present at the bargaining table. While this condition seems eminently reasonable in the context of a private good division problem, it is quite inappropriate in ours. We are intensely interested in precisely the synergies and interactions between players that are axiomatized away by the Lensberg condition. For example, suppose that player i has very limited personal bargaining resources. If the remaining major players are organized into two, equally matched factions, then player i might hold the "balance of power" and so wield a great deal of influence.[14] On the other hand, suppose that the balance of power between factions is disrupted by the "exit" of a key member of one faction, say j. If this departure leaves the second faction in a commanding position, then the influence of our minor player might evaporate. The point here is that in many collective decision-making contexts, the leverage that one player can exert in negotiations may depend in a critical way on the configuration of other participants at the bargaining table.

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[13] Another, less significant respect in which our orientation differs from the classical one concerns the role of exit. In private-good division models, it may sometimes be natural to allow individual participants in multilateral negotiations to "take their money and run." In the collective decision-making contexts that concern us--for example, the public good location problem--this kind of incremental resolution of the bargaining problem is, obviously, not feasible.

[14] An obvious example of this phenomenon is the tremendous power wielded by the tiny religious parties in the Israeli parliament.

SECTION 2. THE T-ROUND BARGAINING GAME.

There is a finite set of players, denoted by $I = \{1, \cdots, \bar{\imath}\}$. The representative player will be denoted by i. The players meet together to select a *policy* from some set, X, of *admissible policies.*

Assumption A1: X is a convex, compact subset of l-dimensional Euclidean space.

If the policy vector x is selected, player i receives the payoff $u_i(x)$.

Assumption A2: For each i, $u_i(\cdot)$ is continuous and strictly concave on X^* and satisfies the von-Neumann Morgenstern axioms.[15]

Contrary to the usual practice in the bargaining literature, players' payoffs do not depend on the time it takes to reach agreement.[16] Of the other assumptions we impose on u_i, the only significant one is strict concavity (i.e., players are globally risk averse). We avoid degenerate special cases by imposing the condition that there is a minimal amount of diversity in players' preferences. Specifically:

Assumption A3: For $i = j$, the maximizers of $u_i(\cdot)$ and $u_j(\cdot)$ on X are distinct.

There is in addition to X a distinguished vector, $\varnothing$, which is called the *disagreement outcome.*[17] If players cannot reach an agreement during the negotiation process then the vector $\varnothing$ will be imposed by default. Once again we avoid degenerate special cases by assuming that there exists a negotiable settlement that Pareto dominates the disagreement outcome:

Assumption A4: There exists $x \in X$ such that for each i, $u_i(x) > u_i(\varnothing)$.

Denote by X^* the set $X \cup \{\varnothing\}$. We will refer to the vector-valued function, $\mathbf{u} = (u_i)_{i \in I}$ defined on X^* as the *payoff function* for the game. (Throughout the paper, we will denote vectors by boldface

[15] For many applications, the requirement of strict concavity is too strong. For example, if X is the unit simplex, representing players' shares of a dollar, then we would naturally want to allow player i to be indifferent between any two share vectors whose i'th components are the same. To allow for such preferences, we could assume that for each i, there is some subspace X^i of X such that i is indifferent between any two vectors that differ only on $X - X^i$, and globally risk-averse on X^i. All of the results in the paper go through if Assumption A2 is weakened in this way.

[16] It is straightforward but not particularly illuminating to incorporate time-discounting into the model.

[17] It is convenient to isolate $\{\varnothing\}$ from the set X. For example, we can then assign $\varnothing$ a payoff of negative infinity without violating continuity.

letters.) The main results of the paper concern families of games generated by varying *either* the payoff function *or* the number of negotiating rounds, while holding all other parameters constant. Accordingly, when all other parameters of the game have already been specified we will denote by $\Gamma(\mathbf{u}, T)$ a T-round bargaining game with payoff function $\mathbf{u}$.

Many of the examples that are discussed in the paper belong to a class of games that we call *spatial games*. In such games, the set X of admissible policies consists of alternative *locations*. For example, a location could be a site for a public good. More abstractly, it could be a point in characteristics space, representing, for example, the attributes of a candidate for some office. Each player has an "ideal" location in X, called her *bliss point*. The vector of players' bliss points will be denoted by $\alpha = (\alpha_i)_{i \in I}$. Letting $d(x, y)$ denote the Euclidean distance between x and y, the utility that i derives from a policy x is a declining function of $d(x, \alpha_i)$.[18] In most of the examples discussed below, we assume that player i has a constant relative risk aversion function of the form:

$$u_i(x) = \rho_i^{-1}(\gamma_i - d(x, \alpha_i))^{1-\rho_i}; \quad u_i(\varnothing) = -\infty , \tag{2.1}$$

where γ_i is a constant and $\rho_i \in (0, 1)$ is player i's *risk aversion coefficient*.

As part of the specification of a multilateral bargaining game, there is a list of *admissible coalitions*, $\mathbb{C}$, with representative element C. An admissible coalition is interpreted as a subset of the players that can impose a policy decision on the whole group. For example, to model decision-making by majority rule, we would define a coalition to be admissible if and only if it contained a majority of the group. More generally, the set of admissible coalitions might have a variety of structures. In particular, we will sometimes impose the restriction that one or more players belongs to *every* admissible coalition. Any such player will be referred to as *essential*.

We now describe the formal structure of a T-round game, beginning with the specification of players' strategies. We distinguish between the odd-numbered rounds of the game, which are called

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[18] If $X \subset \mathbb{R}^l$, then $d(x, y) = \left[\sum_{k=1}^{l} (x_k - y_k)^2 \right]^{½}$.

offer rounds, and the even-numbered rounds, which are called *response rounds*. In an offer round, each player chooses a proposal, consisting of a policy from X and a coalition from $\mathbb{C}$. In a response round, each player specifies an *acceptance set*, indicating which vectors i will accept if invited to join a coalition in that round. We impose the restriction that acceptance sets must be closed. For $t \in \{1, 3, \ldots, T-1\}$, let $(x_{i,t}, C_{i,t})$ denote player i's proposal in offer round t, and $A_{i,t+1}$ represent her acceptance set in the following response round.

A *strategy* for player i is a collection of proposals and acceptance sets, $s_i = \left[(x_{i,1}, C_{i,1}),\ A_{i,2}, \cdots, (x_{i,t}, C_{i,t}),\ A_{i,t+1}, \cdots, (x_{i,T-1}, C_{i,T-1}),\ A_{i,T}\right]$. Let S_i denote the set of strategies available to player i. Note that we have restricted strategies to be *history independent*. That is, players' decisions in round t are required to be independent of the history of moves by nature, and of the history of proposals offered and rejected in previous rounds. We will explain below that this assumption is innocuous for "generic" games.[19] Moreover, acceptance sets can be conditioned neither on the identity of the proposer nor on the composition of the proposed coalition.[20]

A *strategy profile* is a list of strategies, one for each player. Let S denote the set of strategy profiles. A list of strategies for all but one player will be called a *subprofile*. Denote by $S_{-i} = \prod_{j \neq i} S_j$ the set of subprofiles that omit player i, with representative element s_{-i}.

We now explain how strategies are mapped to *outcomes*, which are random variables defined on $X^* = X \cup \{\varnothing\}$. This mapping will be referred to as the *outcome function* for the game. In our heuristic description of the model in section I, we attributed the randomness in the model to moves taken by nature between each offer and response round. From a *formal* standpoint, however, there is

[19] Obviously, this assumption is tenable only because players in our game have complete information.

[20] This last assumption is unlikely to cause serious concern to economists, who tend to insist that the variables in question should not matter. To other social scientists and the world at large, however, they may be regarded as seriously restrictive. In a model of Middle East negotiations, for example, it would be unfortunate if Israelis were obliged to respond in the same way to any given proposal, regardless of whether it was issued by, say, the U.S. or the P.L.O. Both conditions can be relaxed without affecting the main results of the paper. Certain properties of equilibrium will be affected, however.

no need to specify an actual sequencing for nature's moves, since we have defined players' strategies to be independent of these moves. We simply define the outcome function to be a map χ from strategy profiles and "proposer sequences" to policies. A *proposer sequence* is a list of players, one for each offer round, denoted by $\iota = (\iota(1), \iota(3), , \cdots, \iota(T-1)) \in I^{T/2}$. The heuristic interpretation of ι is that for $t \in \{1, 3 ...., T-1\}$, if negotiations have not already been concluded by the time round t is reached, "nature" declares that player $\iota(t)$'s round t proposal will be voted upon in round t+1 by the coalition she specifies. For each t, $\iota(t)$ is an i.i.d. random variable, distributed according to the exogenously specified vector of *access probabilities*, $\mathbf{w} = (w_i)_{i \in I} \gg 0$. (Recall that the magnitude of w_i is interpreted as a measure of player i's relative "political" or "bargaining power.") Thus, the proposal sequence ι is selected with probability $\omega(\iota) = w_{\iota(1)} \times w_{\iota(3)} \times \cdots \times w_{\iota(T-1)}$.

Now fix a strategy profile s, where $s_i = (x_{i,t}, C_{i,t}, A_{i,t+1})_{t=1,3,\cdots,T-1}$. For each $\iota \in I^{T/2}$, a unique policy $\chi(\iota, s)$ is defined as follows. If the policy $x_{\iota(1),1}$ is an element of $A_{j,2}$, for every j in $C_{\iota(1),1}$, then this vector is *accepted* and negotiations do not proceed beyond round 2. Now suppose that for $t \in \{3, 5 ...., T-1\}$, the policies proposed in previous offer rounds have all been rejected. If $x_{\iota(t),t}$ is an element of $A_{j,t+1}$, for every j in the coalition $C_{\iota(t),t}$, then this vector is accepted and negotiations do not proceed beyond round t+1. If agreement is not reached by round T, then the vector $\varnothing$ is selected by default.

The procedure just described defines a finite-support random variable on X^*. Given a profile s, we denote by $Eu_i(\mathrm{s})$ player i's expected payoff from the random profile generated by s. That is, $Eu_i(\mathrm{s}) = \sum_{\iota \in I^{T/2}} \omega(\iota)\, u_i(\chi(\iota, \mathrm{s}))$. Similarly, for $t \in \{3, ..., T+1\}$, $Eu_i(\mathrm{s} | t)$ denotes player i's expected payoff if the profile s is played out starting from round t. We will refer to $Eu_i(\mathrm{s} | t)$ as player i's *reservation utility in round* $t-1$, since this is indeed her expected utility conditional on failure to reach agreement in round $t-1$.

The standard solution concept for games of this kind is *subgame perfection.* Informally, a strategy profile is subgame perfect if starting from each round of the game, the remaining portion of each player's strategy is optimal given remaining portions of the strategies chosen by the other players. In the present context, this concept has no predictive power: for any game in which at least two players are required for agreement, any policy that is weakly preferred by all players to the default outcome can be implemented with certainty as a subgame perfect equilibrium outcome. For example, the following strategies implement the policy x with certainty. In each offer round, each player proposes x and an arbitrary coalition; in each response round, each player accepts x and no other policy. If x is preferred by all players to $\varnothing$, then these strategies are clearly subgame perfect and implement x with probability one.

Equilibria of the kind just described violate a natural rationality criterion and can be eliminated by any one of a number of equilibrium refinements. The best known of these is Myerson's [1978] properness criterion. Because strategy sets in the present game are infinite, this criterion involves some technicalities (see Simon and Stinchcombe [1991]). To avoid these, we will use a simpler refinement, which we will call the SEDS criterion (Sequential Elimination of Dominated Strategies).[21]

Informally, the procedure begins by eliminating strategies that involve inadmissible (i.e., weakly dominated) play in the final response round. Next, considering only strategies that survive the first round of elimination, we eliminate strategies that involve inadmissible play in the penultimate round, which is the final offer round. And so on. To define the criterion formally, we need some more definitions. First, every strategy for i is declared to be *admissible from round* $T+1$. Now fix $t \leq T$ and assume that there is an identified set consisting of strategies that are *admissible from round* $t+1$. Define s_i to be *admissible from round* t if (i) it is admissible from round $t+1$ and if there exists no alternative strategy σ_i such that: (ii) σ_i agrees with s_i before t, (iii) σ_i does at least as well as s_i against any subprofile s_{-i} that is admissible from round $t+1$; and (iv) σ_i does strictly better than s_i against some such subprofile. Finally, say that a profile s satisfies the *SEDS criterion* if for each i, s_i

[21] This criterion is a natural extension to sequential games of the criterion known as Dominance Solvability (Moulin [1979]). Binmore-Osborne [1990] use a similar criterion in a two-person bargaining game. Also, Salant-Goldstein [1990] eliminate dominated strategies to deal with a problem closely related to ours. For a rather different application of the same criterion, see Simon-Stinchcombe [1989].

is admissible from round one. If s satisfies the SEDS criterion for some bargaining game, we say that s is an *equilibrium* for that game. We will refer to the outcome generated by s as an *equilibrium outcome* for the game.

Proposition I below characterizes the set of strategy profiles that satisfy the SEDS criterion. Indeed, the characterization theorem provides the basis for our computer algorithm for solving bargaining games numerically. In each round of the game, after strategies that are inadmissible from later rounds have been eliminated, each player is left with a straightforward single-person decision problem. In a response round, a player will accept a proposed policy if and only if it generates at least as much utility as her reservation utility in that round.[22] In an offer round, a player is faced with a two-part problem. For each admissible coalition, she maximizes her utility on the constraint set defined by other coalition members' reservation utilities in the following round. She then selects the maximizer of these maximizers.

Proposition I: A profile s is an equilibrium for a game $\Gamma(\mathbf{u}, T)$ satisfying Assumptions A1-A4 if and only if for each i and each $t \in \{1, 3, \dots, T-1\}$:

(i) $A_{i,t+1} = \{x \in X : u_i(x) \geq Eu_i(\mathrm{s}|\, t+2)\}$.

(ii) $x_{i,t} \in A_{j,t}$, for all $j \in C_{i,t}$ and $x_{i,t}$ maximizes $u_i(\cdot)$ on the set $\bigcup_{C \in \mathbb{C}} \bigcap_{j \in C} \{x \in X : u_j(x) \geq Eu_j(\mathrm{s}|\, t+2)\}$.

The proposition is proved in the Appendix. The proof depends on two independently useful properties of equilibria, stated in the Lemma below. First, at least two distinct offers are proposed in every offer round. Second, in every offer round there is some policy that yields each player strictly more utility than her reservation utility in the following round.

22 Our assumption that acceptance sets are closed resolves the indeterminacy that arises when players are indifferent between accepting and rejecting a proposal.

Lemma I: Let s be an equilibrium for a game $\Gamma(\mathbf{u}, T)$ satisfying Assumptions A1-A4, $s_i = (x_{i,t}, C_{i,t}, A_{i,t+1})_{t=1,3,\cdots,T-1}$. Then for $t \in t=1, 3, \cdots, T-1$,

(a) There exist distinct players i and j such that $x_{j,t} \neq x_{i,t}$.

(b) There exists $x \in X$ such that for all i, $u_i(x) > Eu_i(\mathbf{s} \mid t+2)$.

An obvious corollary of Proposition I (indeed, of Lemma I(b)), is that in every game, agreement is reached immediately with probability one. We will exploit this fact to obtain a convenient, simplified representation of equilibrium outcomes. Given an equilibrium strategy profile s, we will denote by $\mathbf{x}(\mathbf{s}) = (x_i(\mathbf{s}))_{i \in I}$ the vector consisting of the policies proposed by each player in the first round of negotiations. As we have noted, each of these proposals is necessarily accepted. Therefore, $\mathbf{x}(\mathbf{s})$ is an enumeration of the support of the outcome generated by s. For this reason, we will refer to $\mathbf{x}(\mathbf{s})$ as an *equilibrium outcome vector*. By combining $\mathbf{x}(\mathbf{s})$ with the access probability vector, $\mathbf{w}$, in the obvious way, we can recover the original outcome: for each i, $x_i(\mathbf{s})$ is realized with probability $\sum_{\{j: x_j(\mathbf{s}) = x_i(\mathbf{s})\}} w_j$.

Another corollary of Proposition I is that we can without loss of generality restrict attention to games in which the set of coalitions is minimal in the following sense. Say that a coalition C is *minimal with respect to player i* if there exists no strict subset, C' of C such that the coalition $C' \cup \{i\}$ is admissible.[23] Corollary I below shows that that player i's opportunity set is unaffected by the restriction that she must choose only coalitions which are minimal with respect to i. In other words, we lose no generality by assuming that i always chooses coalitions that (i) include herself whenever possible, and (ii) exclude as many other players as possible. This fact is of considerable practical importance, because when we use numerical methods to study games, it is obviously important to minimize the the number of coalitions for which calculations must be made.

[23] Note that our criterion is strictly more stringent than the simpler criterion of (unqualified) minimality, which would be satisfied by any coalition rendered inadmissible by the ommission of any player. For example, in a majority rule game with five players, the coalition $\{2, 3, 4\}$ is admissible, but is *not* admissible with respect to player #1, since $\{1, 3, 4\}$ is admissible.

Corollary to Proposition I: Let s be an equilibrium for a game $\Gamma(\mathbf{u}, T)$ satisfying Assumptions A1-A4, where $s_i = (x_{i,t}, C_{i,t}, A_{i,t+1})_{t=1,3,\cdots,T-1}$. Then there is an equilibrium profile, σ, for this game which is identical to s with the (possible) exception that in each round, each player i specifies a coalition that is minimal with respect to i.

We can now state the main result of this section. An immediate implication of Proposition I is that an equilibrium always exists. Moreover, for generic games the equilibrium outcome is unique. Specifically, let $\mathbb{U}$ denote the set of payoff functions on X satisfying Assumptions A2-A4 and endow $\mathbb{U}$ with the sup norm metric.[24] We now have:

Theorem II: Every T-round bargaining game satisfying assumptions A1-A4 has an equilibrium. Moreover, there is an open, dense subset, $\mathbb{U}'$, of $\mathbb{U}$ such that for each $\mathbf{u}' \in \mathbb{U}'$ and every T, the equilibrium outcome for $\Gamma(\mathbf{u}', T)$ is unique.

The arguments we use to prove uniqueness also show that except in exceptional games, history-independence is an innocuous assumption. The argument is very straightforward. In each round of the game, players' payoffs and opportunity sets are independent of anything that has happened in previous rounds. Also, because there is no uncertainty about players' types in the model, the unfolding of history cannot reveal any payoff relevant information. Obviously, whenever a player has a unique optimal choice, and this choice is independent of history, the player must act in the same way, regardless of the past history. Finally, in the present context it is generically the case that players' optimal choices are indeed unique in every round.

[24] In the sup norm metric, the distance between two functions is the supremum, taken over all points x in the domain, of the absolute value of the difference between the functions evaluated at x.

SECTION 3. THE MULTILATERAL BARGAINING MODEL.

A *multilateral bargaining model* consists of a sequence of *T-round bargaining games*, $\{\Gamma(\mathbf{u}, T)\}_{T=2,4,\cdots}$ in which T increases without bound. The games in the sequence are all variants of the same *underlying game;* the only thing that changes is the number of negotiating rounds. Modifying traditional notation somewhat, we will denote the underlying game by $\Gamma(\mathbf{u}, \cdot)$, without specifying the number of rounds. We will then consider the multilateral bargaining model that is *derived* from this game.

The equilibrium outcomes for games in the sequence are probability measures on the set of admissible policies. We define a *solution* to our model to be a limit of a sequence of equilibrium outcomes for the games in the sequence. Since the outcomes are random variables, the natural topology in which to take limits is the weak-star topology.[25] Because equilibrium outcomes have a special structure, however, we can simplify matters considerably: it is sufficient simply to identify the pointwise limits of sequences of *equilibrium outcome vectors.* Specifically, suppose that for $\tau = \{2, 4, \cdots,\}$, $\mathbf{s}^\tau$ is an equilibrium strategy profile for the game $\Gamma(\mathbf{u}, \tau)$ and that $\bar{\mathbf{x}} = (\bar{x}_i)_{i\in I}$ is a pointwise limit of the sequence $(\mathbf{x}(\mathbf{s}^\tau))_{\tau=2,4,\cdots,}$. We will refer to $\bar{\mathbf{x}}$ as a *limit outcome vector.* Then from our earlier discussion, it will immediately be clear that the *outcomes* generated by $(\mathbf{s}^\tau)_{\tau=2,4,\cdots,}$ have a weak-star limit, which is defined by combining $\bar{\mathbf{x}}$ with $\mathbf{w}$ as before: for each i, $\bar{x}_i$ is realized with probability $\sum_{\{j:\bar{x}_j=\bar{x}_i\}} w_j$.

We say that a solution is *deterministic* if the limit outcome has singleton support, or, equivalently, if the elements of the the limit outcome vector are all identical. We say that a deterministic solution *implements* the vector to which it assigns probability one. Solutions that are not deterministic will be called *stochastic.* When a solution exists, it is interpreted in the usual way, as a proxy for the equilibrium outcome of the underlying game when the number of negotiation rounds is arbitrarily large.

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[25] See, for example, Parthasarathy [1967].

It is of primary importance to identify conditions under which a solution exists. In a model that has no solution, the outcome of negotiations depends in a nontrivial way on the number of negotiation rounds. In such instances, the model can have little predictive power or positive prescriptive interest. Existence failures are nonetheless interesting in a negative sense: they can be interpreted as evidence of inherent instabilities in the negotiating environment.

The main results of this paper identify conditions under which deterministic solutions exist. A necessary but not sufficient condition for existence of a deterministic solution is that the underlying game has a nonempty *core*. The core of a multilateral bargaining game is defined in the obvious way. A policy x can be *blocked* by coalition C if there exists an alternative policy y such that each member of C strictly prefers y to x. The core consists of policies that cannot be blocked by any admissible coalition. The necessity result is presented as Proposition III below. Since the proof is both simple and instructive, we include it in the text.

> Proposition III: Let $\Gamma(\mathrm{u}, \cdot)$ be a multilateral bargaining game satisfying assumptions A1-A4. If the multilateral bargaining model derived from this game, $\{\Gamma(\mathrm{u}, T)\}_{T=2,4,\cdots}$, has a deterministic solution, then the policy implemented by this solution belongs to the core of $\Gamma(\mathrm{u}, \cdot)$.

Proof of Proposition III: Assume that x is implemented but that there is some policy y and some admissible coalition C such that each member of C strictly prefers y to x. Then there exists $\varepsilon > 0$ such that all members of C strictly prefer y to any policy in the ball $B(x, \varepsilon)$. For $\tau = 2, 4, \cdots$, let s^{τ} be an equilibrium profile for $\Gamma(\mathrm{u}, \tau)$. For τ sufficiently large, each component of the equilibrium outcome vector $x(s^{\tau})$ must be contained in $B(x, \varepsilon)$. Thus we have $u_j(y) > u_j(x_j(s^{\tau})) > Eu_j(s_j^{\tau} | 3)$, for every $j \in C$. (The second inequality follows from combining Proposition I(ii) and Lemma I(b).) But this is a contradiction, since by Proposition I(ii), $x_i(s^{\tau})$ must a maximizer of $u_i(\cdot)$ on the set $\bigcap_{j \in C} \{x \in X : u_j(x) \geq Eu_j(s^{\tau} | 3)\}$. $\square$

The following three-player spatial game $\Gamma(\mathrm{u}, \cdot)$. illustrates what can happen when the core of the game is empty. The three players' utility functions are as defined in equation (2.1) above, with bliss points that form a triangle in $\mathbf{R}^2$: $\alpha_1 = (-1,0)$, $\alpha_2 = (+1,0)$ and $\alpha_3 = (0,1)$. For each i, set $\rho_i = 0.5$ and $w_i = 1/3$. Assume majority rule, so that any coalition containing two players is admissible. To see that the core of this game is empty, pick any point, $x = (x_1, x_2)$, in the convex hull of the α_i's (see Figure 3.1(a)). (Note that in this context, the subscripts on x denote components of the policy rather than player indices.) Assume without loss of generality that $x_2 > 0$. Clearly, players #1 and #3 strictly prefer the projection of x onto the horizontal axis to x. Therefore, the coalition consisting of these two players can block x.

The sequence of outcomes for T-round games has a limit cycle, illustrated in Figure 3.1(b). There exists T sufficiently large such that for $\tau \in \{T+2, T+6, \cdots\}$, each player includes player #1 in the coalition she proposes in the first round of $\Gamma(\mathbf{u}, \tau)$ and, consequently, the outcome of this game generates a slightly higher expected utility for player #1 than for #3. For $T \in \{\bar{T}+4, \bar{T}+8, \cdots\}$, the situation is reversed: each player includes player #3 in the coalition she proposes in the first round of $\Gamma(\mathbf{u}, \tau)$ and, consequently, the outcome of this game generates a slightly higher expected utility for player #3 than for #1. The explanation for this oscillatory pattern readily becomes apparent by comparing the $(T+2)$-round and $(T+4)$-round games. The proposals made in the first round of the former game are the same as the proposals made in the third round of the latter. In the $(\bar{T}+4)$-round game, player #1 has a higher expected utility than #3 conditional on reaching the third round. In round #1 of this game, therefore, player #1 takes a tougher bargaining stance than #3, and because of this is a relatively unattractive coalition partner. (Alternatively, her reservation utility is too high.) But in round #1 of the $(\bar{T}+6)$-round game, the positions are reversed and it is player #3 whose reservation utility is too high.

Our next example establishes that nonemptiness of the core is not sufficient to guarantee existence of a deterministic solution. Consider a four-person spatial game in which players' bliss points form a square; $\alpha_1 = (+1,+1)$, $\alpha_2 = (+1,-1)$, $\alpha_3 = (-1,-1)$ and $\alpha_4 = (-1,+1)$. In this example, players' utilities are construct explicitly. They do not belong to the class of utilities characterized by equation (2.1). Construct $u_1(\cdot)$ so that its level sets are circles centered at α_1, with $u_1(\alpha_1) = 4$,

$u_1(\alpha_2) = u_1(\alpha_4) = 0$, $u_1(\alpha_3) = -8$, while $u_1(\emptyset) < -8$. Define the other players' utilities symmetrically. For each i, set $w_i = 0.25$ and assume that any three-person coalition is admissible.

It is straightforward to verify that the core of this game is the singleton set consisting of the origin. On the other hand, the bargaining model that corresponds to this game has a unique, stochastic solution, in which the limit outcome vector is $\bar{\mathbf{x}}$, with $\bar{x}_i = \alpha_i$ for each i. To see this, observe that in the last offer round, each player proposes her bliss point; if selected by nature, this proposal will be accepted in the next round. Now consider a nonterminal response round and assume that in the offer round that follows, each player will proposes her bliss point. Thus, each player's reservation utility in this round must be -1. Therefore, players #2 and #4 will accept α_1 if it is proposed, and #1 will indeed propose α_1 in the preceding offer round. Since the game is symmetric, we have established that in each round, each player proposes her bliss point and all proposals are accepted.

Theorems IV and V below identify two sets of sufficient conditions for existence of a deterministic solution. The first condition is that the space of policies X is one dimensional and that decisions are made by majority rule.

> Theorem IV: Let $\Gamma(\mathbf{u}, \cdot)$ be a multilateral bargaining game satisfying assumptions A1-A4. If (i) the space of admissible policies, X, is a subset of $\mathbf{R}^1$ and (ii) a coalition is admissible if and only if it contains strictly more than half of the players in I, then the multilateral bargaining model derived from $\Gamma(\mathbf{u}, \cdot)$ has a deterministic solution.

When the policy space, X, is multidimensional, the problem of establishing convergence is much more difficult. One way to guarantee convergence is to assume that the game has at least one "essential player," i.e., a player who is a member of every admissible coalition. The interpretation of this assumption was discussed in detail above (pp. 5-6).

Theorem V: Let $\Gamma(\mathbf{u}, \cdot)$ be a multilateral bargaining game satisfying assumptions A1-A4. If there is at least one essential player, then the multilateral bargaining model derived from $\Gamma(\mathbf{u}, \cdot)$ has a deterministic solution.

An immediate corollary of Theorem V is that every *unanimity game* has a deterministic solution. In such games, there is only one admissible coalition, and all players are essential.

Our final result is an immediate consequence of Theorem II: solutions, when they exist, are generically unique. Once again, let $\mathbb{U}$ denotes the set of payoff functions on X satisfying Assumptions A2-A4.

Corollary to Theorem II: Let $\Gamma(\cdot, \cdot)$ be a family of multilateral bargaining games satisfying assumptions A1-A4, in which the payoff function is drawn from $\mathbb{U}$. There is an open, dense subset, $\mathbb{U}'$, of $\mathbb{U}$ such that for each $\mathbf{u}' \in \mathbb{U}'$, if the model derived from $\Gamma(\mathbf{u}', \cdot)$ has a solution, then this solution is unique.

Proof of the Corollary: Suppose that for some $\mathbf{u} \in \mathbb{U}$, the model derived from $\Gamma(\mathbf{u}, \cdot)$ has more than one solution. Then necessarily there exists T (in fact, infinitely many T's) such that the bargaining game $\Gamma(\mathbf{u}, T)$ has at least two distinct equilibrium outcomes. But from Theorem II, it follows that the set of all such $\mathbf{u}$'s is contained in the complement of an open, dense subset of $\mathbb{U}$. □

SECTION 4. EXAMPLES AND APPLICATONS

The main purpose of the section is to illustrate properties of the model and to suggest contexts in which our model might be usefully applied. In some of the examples, the core of the underlying game contains a single element. In these cases, the solution to the bargaining model is obtained immediately and is insensitive to changes in the configuration of negotiators' bargaining attributes. In the other examples, our model provides a procedure for selecting from among the multiple elements of the core. In these cases, the selection procedure is sensitive to players' bargaining attributes, and it is instructive to study the comparative statics properties of the model. In our discussions of comparative statics, we will consistently refer to the original parameter set as the *base-case scenario;* once a parameter has been changed, we will refer to the *modified scenario.* Throughout this section, our discussion will be heuristic and informal. For a more formal presentation of these and related comparative statics results, see Rausser-Simon [1991c].

Examples #1-#4 are spatial models. In each case, we assume that the space of policies, X, is a subset of two-dimensional Euclidean space. Players' preferences belong to the family defined in equation (2.1). In our base-case scenarios, we always assume strict majority rule. One possible interpretation, applicable to all four examples, is that the players comprise the executive commmittee of a political party, who are negotiating over the details of the party's official political platform. The first component of a player's bliss point indicates whether the player is a radical or a conservative. Players whose bliss points are to the left (resp. right) of the origin represent the "left-wing" (resp. right-wing) of the party. Bliss points located further from the origin represent more extreme political orientations. The vector of access probabilities can be thought of as reflecting the distribution of political influence among committee members. Agents' risk aversion coefficients can be interpreted either literally, or more metaphorically as a measure of the extent to which the agent is a "tough" negotiator.

Example 1: Trilateral bargaining when one or more players are essential.

For our first example, we revisit the three-person, majority rule game introduced on p. . Recall that this game had three players, whose bliss points, respectively, were: $\alpha_1 = (-1,0)$, $\alpha_2 = (+1,0)$ and $\alpha_3 = (0,1)$. Though this game is extremely simple, it can be varied in several ways to illustrate a variety of important facets of our framework.

As we have already seen, without at least one essential player, this game has no solution. Now consider the effect of declaring one or more players to be essential. First assume the minimal admissible player coalitions are (1, 2) and (1, 3), so that player #1 is essential.[26] From Theorem V, there is in this case a deterministic solution. Unfortunately, however, the solution is not very

[26] The grand coalition is admissible but is not minimal with respect to any player.

interesting: it simply implements player #1's bliss point, $\alpha_1 = (-1, 0)$. To verify that this is the unique solution, observe first that α_1 is contained in the core of the underlying game. Indeed, in any game, any essential player's bliss point is contained in the core. In this particular game, the core contains no other policy. It therefore follows from Theorem II that the unique solution must implement α_1.

Next, consider the unanimity version of this game, in which all three players are essential. In this case, the core of the underlying game consists of the convex hull of the three bliss points. The solution now depends entirely on the distribution of bargaining attributes among the three players. The comparative statics properties are predictable: in response to an increase in one player's access probability or a reduction in her risk aversion coefficient, the solution will shift in the direction of her bliss point.

From a heuristic perspective, the most revealing version of the game has two essential players. For example, suppose that the admissible coalitions are (1, 2) and (1, 2, 3), so that #1 and #2 are essential, while #3 is not. In this case, clearly, the core of the underlying game is the "contract curve" between players #1 and #2, which in this case is the line segment joining their bliss points. If both players have equal access, the solution of the model will implement the midpoint of this line, i.e., the origin.

It is instructive to investigate the role of player #3 in this version of the problem. Clearly, players #1 and #2 will each propose the coalition (1, 2), so that #3 will participate in a coalition only if she herself is the proposer. Provided her access probability is positive, however, her presence at the bargaining table will still have an impact on the negotiations.. To see this, we will consider the comparative statics effects of a shift in her bliss point from α_3 to, say, $\alpha'_3 = (-\varepsilon, 1)$. In the simulation, we set $\rho_i = 0.2$, for each i and assume each player has equal access. The constant γ_i in equation (2.1) is set equal to 100. Table 4.1 summarizes a numerical simulation of the effect of this change, for $\varepsilon = 0.05$. We display the proposals made in the last three offer rounds, in both the base-case and modified scenarios. The first three lines in each block list the policies proposed by each player, followed by the utilities derived by each of the players from these offer. An asterisk beside a utility level indicates that this level is equal to the player's reservation utility in the following response round, so that meeting this reservation utility is a binding constraint on the proposer. The fourth line displays each player's expected utility conditional on reaching the round; these numbers are also the reservation utilities of the players in the preceding response rounds.

Consider the modified scenario. In the final offer round (i.e. $T-1$), #3 will now propose α'_3, a change that will benefit #1 and hurt #2. In the preceding response round, therefore, #1's reservation utility will be higher than before, while #2's will be lower. Now consider the penultimate offer round. Relative to the base-case scenario, there are three changes in this round, all of which benefit #1 at the expense of #2. Player #1's proposal will be closer to α_1, because #2's reservation utility is lower. Player #2's proposal will be further from α_2, because #1's reservation utility is higher. Finally, player #3's proposal will be closer to α_1 and further from α_2, *both* because α'_3 is closer to α_1 than α_3, *and* because of the changes in the reservation utilities of #1 and #2. In this penultimate offer round, then, all of the individual effects are mutually reinforcing, so that relative to the base-case scenario, player #1's expected utility conditional on entering this round will unambiguously increase, while #2's will decrease. Now proceed by backward induction to the first round of negotiations, and conclude that the solution to the model shifts to the left along the horizontal axis.

Example 2: One-dimensional set of bliss points, an odd number of players.

In this and the remaining examples, we assume that the coalition structure is symmetric. In particular, there will be no essential players. Assume that there are $2n + 1$ players and that a coalition is admissible only if it contains at least $n+1$ players. Let $(\alpha_{i,1}, \alpha_{i,2})$ denote player i's bliss point. For each i, assume that the second component of i's bliss point, $\alpha_{i,2}$, is zero, and that players are ordered so that for $i < j$, $\alpha_{i,1} \leq \alpha_{j,1}$, with $\alpha_{n+1,1} = 0$. In this case, the core of the underlying game contains exactly one point: the origin.[27] From theorems II and IV, any game of this kind has a unique,

[27] To see this, observe that players #1 to #$n+1$ strictly prefer the origin to any policy x with x_1 negative. Similarly, players #$n+1$ to #$2n+1$ strictly prefer the origin to any policy x with x_1 positive. Note that the solution to the model is completely invariant to *any* of the parameters of the model, except the location of $\alpha_{n+1,1}$.

deterministic solution, which implements the origin. Thus, our multilateral bargaining framework provides a formal foundation for the familiar "median voter" model.

Example 3: One-dimensional policy space and an even number of players.

This example is identical to the preceding one except that there are $2n$ players instead of $2n+1$. As before, assume that a coalition must contain at least $n+1$ players to be admissible. When there is an even number of players, the core of the underlying game is the interval, $[\alpha_{n,1}, \alpha_{n+1,1}]$, along the horizontal axis between the two median players' bliss points. In contrast to example 2, the solution in this case depends on all of the parameters of the model, so that the comparative statics analysis is much richer. To illustrate some of the model's properities, we will discuss rather informally two of the more subtle comparative statics results.

We begin by examining the effect of an outward shift in the bliss point of the extreme left-wing or right-wing player (i.e., a reduction in $\alpha_{1,1}$ or an increase in α_{2n}). Intuitively, such a change can be interpreted as reflecting a polarization of the party or an increase in political extremism. A change of this kind has two effects, which we will refer to as the *risk aversion* effect and the *access* effect. The risk aversion effect benefits the wing of the party whose member's bliss point has changed; the access effect benefits the wing that has greater aggregate power. Whenever players' utilities are defined by equation (2.1), the former effect is very weak relative to the latter. Hence, if the distribution of access is virtually uniform, the solution will shift in the direction of the original bliss point change. If the distribution is only slightly skewed to the right or left, however, then the more powerful wing of the party will benefit at the expense of the other wing.

To simplify the analysis, we assume that in the base-case scenario, all players are identical, and that bliss points are distributed symmetrically about the origin (i.e., for each $k < n$, $\alpha_{n-k,1} = -\alpha_{n+1+k,1}$). Table 4.3a summarizes a numerical simulation of this case, with $\rho_i = 0.2$, for each i. Now consider the effect of an increase in $\alpha_{2n,1}$. Because player #$2n$ proposes her bliss point in the last offer round (round $T-1$), the effect of this shift is to reduce all other players' expected utilities, conditional on reaching this round. Because players are all equally risk-averse, however, the left-wing will be more seriously affected by the change than the right-wing.[28] Now consider the penultimate offer round (round $T-3$). Because admissible coalitions contain at least $n+1$ members, each left-winger (resp. right-winger) must induce some right-winger (resp. left-winger) to accept her proposal. But as we have observed, the decrease in left-wingers' reservation utilities in round $T-2$ exceeds the corresponding decrease for right-wingers. Thus, while each left-wing proposal in this round will shift to the left, the corresponding shifts in the right-wing proposals will be larger. Conditional on reaching the penultimate offer round, the expected utilities of left-wing players will fall relative to the base-case scenario, while those of the right-wing players will fall by a lesser amount, or possibly even increase. Proceeding by backward induction to the first round, it follows that when players' attributes are all identical, an increase in right-wing extremism results in a shift in the solution to the right.

A variant of the base-case scenario is obtained by transferring some access from the right-wing to the left-wing.[29] Table 4.3b summarizes a numerical simulation of the variant, in which the access probabilities of each of the leftwingers is increased from 0.166 to roughly 0.188. The effect of this modification first becomes apparent in the penultimate offer round. As before, the left-wing proposals shift to the left, and the right-wing proposals shift to the right by a greater amount. However, if the asymmetry in access is sufficiently great, the smaller, but more heavily weighted leftward shift may dominate the larger but less heavily weighted rightward shift, reversing the relative fortunes of the left- and right-wing. Conditional on reaching the penultimate offer round, the expected utilities of the *right*-wingers will fall relative to the base-case scenario, while those of the left-wing players will fall by a lesser amount, or possibly even increase. The reversal in this round will be transmitted back to the first round. Thus in this case, if the left-wing is more powerful than the right, an increase in right-wing extremism may result in a shift in the solution to the left.

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[28] That is, the second derivatives of the u_i's with respect to distance are all equal and negative.

[29] The argument we present below does not depend on the way in which access is transferred.

Another set of comparative statics results focus on the effect of increasing the minimal size of admissible coalitions, assuming that one wing of the party is more powerful than the other. Once again, some restrictions are needed in order to obtain a determinate result. Assume, as before, that bliss points are distributed symmetrically relative to the origin, and that all players are equally risk averse. Moreover, assume that one wing of the party is *uniformly* more powerful than the other, i.e., the access probabilities of each member of one wing exceeds the access probability of every member of the other wing. Under these restrictions, an increase in the size of a minimal coalition unambiguously benefits the more powerful wing of the party.

Example 4: Two-dimensional policy space and an even number of players.

This example is closely related to the preceding one, except that we relax slightly the restriction that players' bliss points lie along the horizontal axis. When the space of policies is two-dimensional, we cannot guarantee in general that a solution exists. To finesse this problem, we confine agents' bliss points to a set that is "nearly" one-dimensional. Players' bliss points are defined in Table 4.2 below.

Table 4.4: Configuration of players' bliss points in Example 4.

Player:	#1	#2	#3	#4	#5	#6
$\alpha_{i,1}$ $\alpha_{i,2}$	$-\theta$ $-\varepsilon_l$	$-\theta$ 0	$-\theta$ $+\varepsilon_l$	$+\theta$ $-\varepsilon_r$	$+\theta$ 0	$+\theta$ $+\varepsilon_r$

Initially, we set $\varepsilon_l = \varepsilon_r = \varepsilon > 0$. A deterministic solution exists whenever θ is sufficiently large relative to ε. We will consider the effect of an increase in ε^r, which is interpreted as a reduction in the internal cohesiveness of the right-wing of the party. Not surprisingly, this change results in a shift to the left in the solution.[30] Intuitively, the loss of cohesion within the right-wing weakens its

[30] To be pedantic, we should add that if the left-wing's access is of order $o(\varepsilon)$), the shift could be in the reverse direction.

bargaining power relative to the left-wing and this translates into a deterioration in the right-wing's performance. Table 4a summarizes a numerical simulation of this case, with $\theta = 9$. Initially, $\varepsilon_l = \varepsilon_r = 1$. In the modified scenario, ε_r is increased to 1.5. In this simulation, players have equal access and ρ_i is set equal to 0.5, for each i.

Clearly, there is a striking difference between this result and the consequence of one-dimensional dispersion that emerges from Example 3. The source of the difference is that in Example 3, the dispersion within the right-wing also affected the left-wing. In the present case, dispersing the right-wing in the "vertical direction" scarcely affects the left-wing at all (assuming that θ is sufficiently large relative to ε). The logic of the argument is quite straightforward. In the last round of proposals, each right-wing player proposes her bliss point. When these points are dispersed, the increase in the variance of the outcome is significant for each of the right-wingers, but insignificant for the left-wingers, so that in the penultimate round of voting, the right-wingers' participation constraints are relaxed relative to those of the left-wingers. The left-wingers' advantage is now transmitted backwards through the game tree in the usual way.

Example 5: A two-good pure exchange economy with four players.

In this final example our framework is applied to a pure exchange economy. While the example is extremely simple, it suggests that the domain of applications is much wider than the previous examples might suggest. Moreover, it represents a departure from the preceding analysis in two notable respects. First, we obtain a deterministic solution even though there is no essential player and the space of policy vectors is of high dimension. This implies that results much more general than theorems IV and V can eventually be obtained. Second, we extend one of the basic assumptions of the underlying model, by allowing players' admissible policy sets to vary, depending on which coalitions they select.

There are two commodities and four players. Assume that each player has equal access. There are no restrictions on the set of admissible coalitions: any combination of players is admissible.

Players #1 and #2 are each endowed with 2 units of the first commodity while players #3 and #4 are each endowed with 2 units of the second. A policy vector is an *allocation* $\mathbf{x} = (x_{i1}, x_{i2})_{i=1}^{4}$ such that for $k = 1, 2$, $\sum_{i=1}^{4} x_{ik} = 4$. An allocation is admissible for a coalition C if each player who is excluded from C is assigned her initial endowment. The default outcome is defined to be the autarky allocation, i.e., each player is assigned her initial endowment. Let z_i denote player i's *net trade*. Player i's utility function is $u_i(\mathbf{z}) = (z_{i1}z_{i2})^{\rho_i}$, where ρ_i must be less than ½ to ensure concavity.

This game has a unique deterministic solution. Not surprisingly, the equilibrium outcome is the equal division allocation: each player receives one unit of each commodity. The proof is briefly sketched below. If player i is selected by nature in round $T-1$ of the T-round game, she will propose the grand coalition and the allocation assigning her the economy-wide endowment, i.e., (4, 4). This allocation is accepted in round T. Assuming equal access, then, each player's expected utility conditional on entering round $T-1$ is ½. Now consider player #1's choice of coalition in round $T-3$. Obviously, she will not form a two-person coalition with player #2. Moreover, her opportunities in a two-person coalition with either #3 or #4 are dominated by her opportunities in the grand coalition: whatever she can obtain in the two-person coalition, she can do better by including the other two players, offering them each the vector (¼, ¼) and keeping the remainder of their joint endowment. A similar argument shows that she prefers the grand coalition to any coalition of three players. It follows, then, that if player i is selected by nature in round $T-3$, she will propose the grand coalition and the allocation which assign (¼, ¼) to each of the other players and the remainder to herself. Now proceed by induction in the obvious way.

The comparative statics of this example are extremely straightforward. Player i's utility from the equilibrium outcome increases as her access probability increases and as ρ_i increases. Note that while the solution implemented the competitive equilibrium allocation in the initial symmetric example, this is no longer the case when symmetry is abandoned. A striking feature of this example is that unlike our earlier examples, there are no synergies between agents. Other things being equal, player i's utility from the solution depends entirely on her own access probability and on ρ_i, and not the distribution of these parameters among the other players.

Table 4.1: Comparative Statics of shifting $\alpha_{3,1}$ to the left in Example 1.

Round	$\alpha_{3,1} = 0$	x_1	x_2	#1's utility	#2's utility	#3's utility
T-1	#1's offer	-1.000	0.000	66.289	66.056	66.124
	#2's offer	1.000	0.000	66.056	66.289	66.124
	#3's offer	0.000	1.000	66.124	66.124	66.289
	Exp utility			66.156	66.156	66.179
T-3	#1's offer	-0.138	0.000	66.189	66.156*	66.171
	#2's offer	0.138	0.000	66.156*	66.189	66.171
	#3's offer	0.000	0.543	66.156*	66.156*	66.236
	Exp utility			66.167	66.167	66.193
T-5	#1's offer	-0.046	0.000	66.178	66.167*	66.172
	#2's offer	0.046	0.000	66.167*	66.178	66.172
	#3's offer	0.000	0.307	66.167*	66.167*	66.208
	Exp utility			66.171	66.171	66.184
Round	$\alpha'_{3,1} = -0.05$	x_1	x_2	#1's utility	#2's utility	#3's utility
T-1	#1's offer	-1.000	0.000	66.289	66.056	66.128
	#2's offer	1.000	0.000	66.056	66.289	66.120
	#3's offer	-0.050	1.000	66.128	66.120	66.289
	Exp utility			66.158	66.155	66.179
T-3	#1's offer	-0.150	0.000	66.190	66.155*	66.172
	#2's offer	0.126	0.000	66.158*	66.187	66.171
	#3's offer	-0.013	0.544	66.158*	66.155*	66.236
	Exp utility			66.169	66.166	66.193
T-5	#1's offer	-0.057	0.000	66.179	66.166*	66.173
	#2's offer	0.034	0.000	66.169*	66.177	66.172
	#3's offer	-0.012	0.307	66.169*	66.166*	66.208
	Exp utility			66.172	66.169	66.184

Table 4.3b: Comparative Statics of shifting $\alpha_{6,1}$ to the right in Example 3: Left-wing has more access.

Rnd	$\alpha_{6,1}=4$	x_1	$u_1(\cdot)$	$u_2(\cdot)$	$u_3(\cdot)$	$u_4(\cdot)$	$u_5(\cdot)$	$u_6(\cdot)$
T-1	#1's offer	-4.000	6.310	5.800	5.278	3.031	2.408	1.741
	#2's offer	-3.000	5.800	6.310	5.800	3.624	3.031	2.408
	#3's offer	-2.000	5.278	5.800	6.310	4.193	3.624	3.031
	#4's offer	2.000	3.031	3.624	4.193	6.310	5.800	5.278
	#5's offer	3.000	2.408	3.031	3.624	5.800	6.310	5.800
	#6's offer	4.000	1.741	2.408	3.031	5.278	5.800	6.310
	Exp utility		4.317	4.688	4.848	4.564	4.303	3.873
T-3	#1's offer	-1.330	4.921	5.451	5.969	4.564*	4.008	3.432
	#2's offer	-1.330	4.921	5.451	5.969	4.564*	4.008	3.432
	#3's offer	-1.330	4.921	5.451	5.969	4.564*	4.008	3.432
	#4's offer	0.806	3.736	4.301	4.848*	5.699	5.175	4.638
	#5's offer	0.806	3.736	4.301	4.848*	5.699	5.175	4.638
	#6's offer	0.806	3.736	4.301	4.848*	5.699	5.175	4.638
	Exp utility		4.406	4.951	5.482	5.058	4.515	3.956
T-5	#1's offer	-0.421	4.427	4.970	5.499	5.055	4.514	3.956*
	#2's offer	-0.421	4.427	4.970	5.499	5.055	4.514	3.956*
	#3's offer	-0.421	4.427	4.970	5.499	5.055	4.514	3.956*
	#4's offer	-0.383	4.406*	4.950	5.479	5.075	4.534	3.977
	#5's offer	-0.383	4.406*	4.950	5.479	5.075	4.534	3.977
	#6's offer	-0.383	4.406*	4.950	5.479	5.075	4.534	3.977
	Exp utility		4.418	4.961	5.490	5.064	4.523	3.965
Rnd	$\alpha_{6,1}=4.4$	x_1	$u_1(\cdot)$	$u_2(\cdot)$	$u_3(\cdot)$	$u_4(\cdot)$	$u_5(\cdot)$	$u_6(\cdot)$
T-1	#1's offer	-4.000	6.310	5.800	5.278	3.031	2.408	1.456
	#2's offer	-3.000	5.800	6.310	5.800	3.624	3.031	2.148
	#3's offer	-2.000	5.278	5.800	6.310	4.193	3.624	2.786
	#4's offer	2.000	3.031	3.624	4.193	6.310	5.800	5.066
	#5's offer	3.000	2.408	3.031	3.624	5.800	6.310	5.592
	#6's offer	4.400	1.456	2.148	2.786	5.066	5.592	6.310
	Exp utility		4.275	4.650	4.813	4.533	4.273	3.663
T-3	#1's offer	-1.386	4.951	5.481	5.998	4.533*	3.976	3.161
	#2's offer	-1.386	4.951	5.481	5.998	4.533*	3.976	3.161
	#3's offer	-1.386	4.951	5.481	5.998	4.533*	3.976	3.161
	#4's offer	0.872	3.698	4.264	4.813*	5.733	5.210	4.455
	#5's offer	0.872	3.698	4.264	4.813*	5.733	5.210	4.455
	#6's offer	0.872	3.698	4.264	4.813*	5.733	5.210	4.455
	Exp utility		4.406	4.952	5.482	5.055	4.513	3.723
T-5	#1's offer	-0.428	4.431	4.974	5.503	5.051	4.510	3.723*
	#2's offer	-0.428	4.431	4.974	5.503	5.051	4.510	3.723*
	#3's offer	-0.428	4.431	4.974	5.503	5.051	4.510	3.723*
	#4's offer	-0.384	4.406*	4.950	5.480	5.074	4.534	3.749
	#5's offer	-0.384	4.406*	4.950	5.480	5.074	4.534	3.749
	#6's offer	-0.384	4.406*	4.950	5.480	5.074	4.534	3.749
	Exp utility		4.420	4.964	5.493	5.061	4.520	3.734

Table 4.4: Comparative Statics of dispersing the right-wing in Example 4.

Rnd	$\alpha_{6,2}=-\alpha_{4,2}=1$	x_1	x_2	$u_1(\cdot)$	$u_2(\cdot)$	$u_3(\cdot)$	$u_4(\cdot)$	$u_5(\cdot)$	$u_6(\cdot)$
	#1's offer	-9.000	-1.000	10.000	9.950	9.899	9.055	9.054	9.049
	#2's offer	-9.000	0.000	9.950	10.000	9.950	9.054	9.055	9.054
	#3's offer	-9.000	1.000	9.899	9.950	10.000	9.049	9.054	9.055
T-1	#4's offer	9.000	-1.000	9.055	9.054	9.049	10.000	9.950	9.899
	#5's offer	9.000	0.000	9.054	9.055	9.054	9.950	10.000	9.950
	#6's offer	9.000	1.000	9.049	9.054	9.055	9.899	9.950	10.000
	Exp utility			9.501	9.510	9.501	9.501	9.510	9.501
	#1's offer	-0.725	-1.000	9.577	9.574	9.565	9.501*	9.499	9.491
	#2's offer	-0.710	0.461	9.570	9.576	9.576	9.496	9.502	9.501*
	#3's offer	-0.725	1.000	9.565	9.574	9.577	9.491	9.499	9.501*
T-3	#4's offer	0.725	-1.000	9.501*	9.499	9.491	9.577	9.574	9.565
	#5's offer	0.710	0.461	9.496	9.502	9.501*	9.570	9.576	9.576
	#6's offer	0.725	1.000	9.491	9.499	9.501*	9.565	9.574	9.577
	Exp utility			9.533	9.537	9.535	9.533	9.537	9.535
	#1's offer	-0.114	-0.867	9.545	9.543	9.535*	9.533*	9.531	9.524
	#2's offer	-0.100	-0.494	9.544	9.544	9.538	9.533*	9.533	9.528
	#3's offer	-0.080	0.871	9.533*	9.541	9.544	9.525	9.533	9.535*
T-5	#4's offer	0.114	-0.867	9.533*	9.531	9.524	9.545	9.543	9.535*
	#5's offer	0.100	-0.494	9.533*	9.533	9.528	9.544	9.544	9.538
	#6's offer	0.080	0.871	9.525	9.533	9.535*	9.533*	9.541	9.544
	Exp utility			9.536	9.538	9.534	9.536	9.538	9.534

Rnd	$\alpha_{6,2}=-\alpha_{4,2}=1.5$	x_1	x_2	$u_1(\cdot)$	$u_2(\cdot)$	$u_3(\cdot)$	$u_4(\cdot)$	$u_5(\cdot)$	$u_6(\cdot)$
	#1's offer	-9.000	-1.000	10.000	9.950	9.899	9.055	9.054	9.046
	#2's offer	-9.000	0.000	9.950	10.000	9.950	9.052	9.055	9.052
	#3's offer	-9.000	1.000	9.899	9.950	10.000	9.046	9.054	9.055
T-1	#4's offer	9.000	-1.500	9.055	9.052	9.046	10.000	9.925	9.849
	#5's offer	9.000	0.000	9.054	9.055	9.054	9.925	10.000	9.925
	#6's offer	9.000	1.500	9.046	9.052	9.055	9.849	9.925	10.000
	Exp utility			9.501	9.510	9.501	9.488	9.502	9.488
	#1's offer	-0.979	-1.223	9.590	9.586	9.575	9.488*	9.484	9.469
	#2's offer	-0.949	0.671	9.580	9.588	9.589	9.477	9.488	9.488*
	#3's offer	-0.979	1.223	9.575	9.586	9.590	9.469	9.484	9.488*
T-3	#4's offer	0.733	-1.270	9.501*	9.497	9.487	9.578	9.573	9.554
	#5's offer	0.722	0.460	9.496	9.501	9.501*	9.565	9.577	9.574
	#6's offer	0.733	1.270	9.487	9.497	9.501*	9.554	9.573	9.578
	Exp utility			9.538	9.542	9.540	9.522	9.530	9.525
	#1's offer	-0.333	-1.241	9.557	9.552	9.542	9.522*	9.518	9.501
	#2's offer	-0.304	-0.725	9.555	9.554	9.546	9.522*	9.522	9.510
	#3's offer	-0.272	1.242	9.539	9.549	9.553	9.504	9.521	9.525*
T-5	#4's offer	0.023	-0.833	9.538*	9.536	9.529	9.539	9.539	9.525*
	#5's offer	0.010	-0.499	9.538*	9.538	9.532	9.537	9.539	9.528
	#6's offer	-0.019	0.911	9.530	9.538	9.540*	9.522*	9.536	9.537
	Exp utility			9.543	9.545	9.541	9.524	9.529	9.521

APPENDIX: PROOFS.

Proof of Proposition I and Lemma I: The proofs of Proposition I and Lemma I are interwoven. We first establish part (i) of the proposition for $t = T$. Consider a policy vector $\bar{x} \in X$ such that $u_i(\bar{x}) < u_i(\emptyset)$. Clearly, if (1) round $T-1$ is reached, (2) some player proposes $\bar{x}$ and (3) i has the deciding vote, then i does strictly worse if she accepts $\bar{x}$ than if she rejects it. Similarly, for x such that $u_i(x) > u_i(\emptyset)$, i does strictly worse if she rejects x than if she accepts it. Moreover, in either case, conditions (1)-(3) are indeed satisfied if each $j \neq i$ plays as follows: $A_{j,T} = X$; $x_{j,T-1} = \bar{x}$; and for each $t \in \{2, 4, \ldots, T-2\}$, $A_{j,t} = \emptyset$. This establishes that if $s_i \in S_{i,T}$, then i's acceptance set in the last period must contain the set $\{x \in X: u_i(x) > u_i(\emptyset)\}$ and exclude the set $\{x \in X: u_i(x) < u_i(\emptyset)\}$. To complete the proof of part (i), observe that acceptance sets are required to be closed.

We now prove parts (a) and (b) of the Lemma, for $t = T-1$. Let $J = \{j \in I: \tilde{x}_{i,T-1} \in \partial\tilde{A}_{j,T-1}\}$. If J is empty, then $\tilde{x}_{i,T-1} \in \text{interior}(\bigcap_{j \in \tilde{C}_{i,T-1}} \tilde{A}_{j,T-1})$ and part (ii) follows immediately from Assumption A3. Assume therefore, that J is nonempty. We will show that for all $j \in J$, $\tilde{x}_{j,T-1} \neq \tilde{x}_{i,T-1}$. It follows from part (i) of Proposition I that for all $j \in J$, $u_j(\tilde{x}_{i,T-1}) = u_j(\emptyset)$. From assumption A4, however, there exists $\bar{x}$ such that $u_{j'}(\bar{x}) > u_{j'}(\emptyset)$, for all $j' \in I$. Since any coalition of players must accept $\bar{x}$ if it is proposed, any player $j \in J$ it follows that $u_j(\tilde{x}_{j,T-1}) \geq u_j(\bar{x}) \geq u_j(\tilde{x}_{i,T-1})$, verifying that as claimed, $\tilde{x}_{i,T-1} \neq \tilde{x}_{j,T-1}$.

For $t = T-1$, part (b) of the Lemma is an immediate implication of Assumption A4. As noted above, for every player i, the vector $\bar{x}$ identified by Assumption A4 will be accepted by all players and yields i a strictly higher payoff than $Eu_i(\tilde{s} \mid T)$.

We now return to the proposition, to prove part (ii) for $t = T-1$. After elimination of weakly dominated strategies in round T, player j is left with a unique admissible choice in round T: the acceptance set $\{x \in X: u_j(x) \geq u_j(\emptyset)\}$. Part (ii) now follows immediately from this fact and part (b) of the Lemma with $t = T-1$.

Now fix $t = \{2, 4, \cdots, T-2\}$ and assume that part (i) of the Proposition has been proved for round $t+2$ while part (ii) of the Proposition and parts (a) and (b) of the Lemma have been proved for round $t+1$. Part (i) of the Proposition can now be proved for t, using exactly the same argument as we used for $t = T$. Now consider parts (a) and (b) of the Lemma, for round $t-1$. If round $t+1$ of the game is reached, then the vector of offers $(\tilde{x}_{i,t+1})_{i \in I}$ will be proposed and accepted. Let $Ex_{t+1} = \sum_{i \in I} w_i \tilde{x}_{i,t+1}$. Because the offers in this round are not all identical, it follows from the strict concavity payoffs that $u_{j'}(Ex_{t+1}) > Eu_{j'}(\tilde{s} \mid 3)$, for every $j' \in I$. Now repeat the argument proving parts (a) and (b) for $t = T-1$, but replace $\bar{x}$ with with Ex_{t+1}. Finally, part (ii) of the proposition for round $t-1$ can be proved by exactly the same argument that was used to prove part (ii) for ound $T-1$. $\square$

Proof of the Corollary to Theorem I: Suppose that for some i and $t \in \{1, 3, \cdots, T-1\}$, $\tilde{C}_{i,t}$ is not minimal with respect to i. Then there exists $C' \subset \tilde{C}_{i,t}$, $C' \neq \tilde{C}_{i,t}$, such that $C' \cup \{i\}$ is admissible. Thus, $\bigcap_{j \in \tilde{C}_{i,t}} \bar{A}_{j,t+1} \subset \bigcap_{j \in C'} \bar{A}_{j,t+1}$. while by Proposition I, $\max\{u_i(\cdot): x \in \bigcap_{j \in \tilde{C}_{i,t}} \bar{A}_{j,t+1}\} \geq \max\{u_i(\cdot): x \in \bigcap_{j \in C'} \bar{A}_{j,t+1}\}$. Since $u_i(\cdot)$ is strictly concave, the maximizers on the two constraint sets must coincide. Moreover, from Lemma I(b) and Proposition I, $u_i(\bar{x}_{i,t}) \in interior(\bar{A}_{i,t+1})$, so that $\bar{x}_{i,t}$ is also a maximizer on $\bigcap_{j \in C' \cup \{i\}} \bar{A}_{j,t+1}$. Thus the profile $\bar{\mathbf{s}}$ remains an equilibrium after substituting the coalition C' for $\tilde{C}_{i,t}$. $\square$.

Proof of Theorem II: Define the set U^* as follows

$$U^* = \{\mathbf{u} \in \mathbb{U} : \text{ there exists } T \text{ and an equilibrium } \mathbf{s} \text{ for } \Gamma(\mathbf{u}, T),$$
$$\text{together with } i \in I,\ 1 \leq t \leq T,\ t \text{ odd, and } x \neq x' \text{ s.t.}$$
$$\text{both } x \text{ and } x' \text{ maximize } u_i^n(\cdot) \text{ on } \bigcup_{C \in \mathbb{C}} \bigcap_{j \in C} \{y \in X: u_j^n(y) \geq Eu_j^n(\mathbf{s} \mid t+2)\}.$$

To prove Theorem II, it is sufficient to show that the closure of U^* has a nonempty interior. Pick $\bar{\mathbf{u}} \in cl(U^*)$ and a sequence $(\mathbf{u}^n)$ in U^* that converges to $\bar{\mathbf{u}}$. We will construct a sequence $(\mathbf{v}^n)$ converging to $\bar{\mathbf{u}}$ such that for all n, $\mathbf{v}^n$ is not contained in U^*. Pick $\bar{\varepsilon} < n^{-1}$ such that every $\mathbf{v} \in B(\mathbf{u}^n, \bar{\varepsilon})$ is a strictly concabold ve function.

We need some additional notation. For each i, if $u_i^n(\cdot)$ has a global maximum on X, let x_i^* denote this point. For each pair of sets $(Y, \Theta) = \{y_i, \{\theta_{i,j}\}_{j \in I}\}_{i \in I}$, where $y_i \in X$ and the $\theta_{i,j}$'s are scalars, define $\pi_i(Y, \Theta)$ by: $\pi_i(Y, \Theta) = \sum_{j \in I} w_j(u_i^n(y_j) + \theta_{i,j})$.

For each j, let $B_{j,T} = (u_j^n)^{-1}([u_j^n(\varnothing), \infty])$. For each i, let $Z_{i,T-1} \subset X$ denote the set of maximizers of $u_i^n(\cdot)$ on the constraint set $\bigcup_{C \in \mathbb{C}} \bigcap_{j \in C} B_{j,T}$. Since for every $C \in \mathbb{C}$, $u_i^n(\cdot)$ has a *unique* maximizer on $\bigcap_{j \in C} B_{j,T}$, $Z_{i,T-1}$ can have only finitely many members. Pick $y_{i,T-1}$ arbitrarily from $Z_{i,T-1}$. Define the sets $Z_{T-1} = \bigcup_i Z_{i,T-1}$ and $Y_{T-1} = \bigcup_i \{y_{i,T-1}\}$. From Lemma I(a), Y_{T-1} contains at least two distinct elements. Also, for each i, j, choose $\theta_{i,j,T-1} \in (-\bar{\varepsilon}, \bar{\varepsilon})$ to satisfy:

(θ-i): $\theta_{i,i,T-1} \geq 0$, with equality if and only if $y_{i,T-1} = x^*$;

(θ-ii): $y_{j,T-1} = y_{j',T-1}$ implies that for all i, $\theta_{i,j,T-1} = \theta_{i,j',T-1}$;

(θ-iii): for $j \neq i$, if $y_{j,T-1} \neq y_{i,T-1}$, then $\theta_{i,j,T-1} \leq 0$, with strict inequality only if $y_{i,T-1} \neq x_i^*$.

(θ-iv): for every i, j and $z \in Z_{j,T-1}$, $\pi_i(Y_{T-1}, \Theta_{i,T-1}) \neq u_i^n(z) + \theta_{i,j,T-1}$, where $\Theta_{i,T-1} = \{\theta_{i,j,T-1}\}_{i,j \in I}$.

Because Y_{T-1} contains at least two distinct points, while $Z_{j,T-1}$ contains only finitely many distinct points, conditions (θ-i,ii,iii) imply that condition (θ-iv) will be satisfied for all but finitely many sets of $\theta_{i,j,T-1}$'s.

Now fix t odd and assume that for each i, the sets Y_{t+2} and Θ_{t+2} have been defined. For each j, let $B_{j,t+1} = (u_j^n)^{-1}([\pi_i(Y_{t+2}, \Theta_{t+2}), \infty])$. For each i, let $Z_{i,t}$ denote the set of maximizers of $u_i^n(\cdot)$ on the constraint set $\bigcup_{C \in \mathbb{C}} \bigcap_{j \in C} B_{j,t+1}$. Pick $y_{i,t}$ arbitrarily from $Z_{i,t}$. Define the sets $Z_t = \bigcup_i Z_{i,t}$ and $Y_t = \bigcup_i \{y_{i,t}\}$. Also, for each i, j, choose $\theta_{i,j,t} \in (-\bar{\varepsilon}, \bar{\varepsilon})$ to satisfy:

(θ-i): $\theta_{i,i,t} \geq 0$, with equality if and only if $y_{i,t} = x^*$;

(θ-ii): $y_{j,t} = y_{j',t}$ implies that for all i, $\theta_{i,j,t} = \theta_{i,j',t}$;

(θ-iii): for $j \neq i$, if $y_{j,t} \neq y_{i,t}$, then $\theta_{i,j,t} \leq 0$, with strict inequality only if $y_{i,t} \neq x_i^*$.

(θ-iv): for every i, j, every $\tau \in \{t, t+2, \cdots, T-1\}$, and every $z \in Z_{j,\tau}$, $\pi_i(Y_t, \Theta_t) \neq u_i^n(z) + \theta_{i,j,t}$, where $\Theta_{i,t} = \{\theta_{i,j,t}\}_{i,j \in I}$.

Once again, because Y_t contains at least two distinct points, while $Z_{j,t}$ contains only finitely many distinct points, conditions (θ-i,ii,iii) imply that condition (θ-iv) will be satisfied for all but finitely many sets of $\theta_{i,j,t}$'s.

Observe that since utilities are strictly concave, $u_i^n(\cdot)$ is locally nonsatiated at every $x \neq x_i^*$. It follows that for each i and each $t \in \{1, 3, 5, \cdots, T-1\}$:

(z-i): if $z \in Z_{i,t}$, then either $z = x^*$ or else $z \in \partial(\bigcap_{j \neq i} B_{j,t})$.[31]

Condition (z-i) together with (θ-iv) imply that for every i, j and $t, \tau \in \{1, 3, \cdots, T-1\}$:

(z-ii): $t \neq \tau$ implies $Z_t \cap Z_\tau \subset \bigcup_i \{x_i^*\}$.

We now construct a payoff function $\mathbf{v}^n$ close to $\mathbf{u}^n$ which has a unique equilibrium in which player i proposes $y_{i,t}$ in period t. First, pick $\delta > 0$ sufficiently small that for any two distinct elements $z, z' \in \bigcup_{\tau \text{ odd}} Z_\tau$,

(δ-i): $cl(B(z, \delta)) \cap cl(B(z', \delta)) = \varnothing$;

Condition (δ-i) can be satisfied because the set $\bigcup_\tau Z_\tau$ is finite.

Next, construct a function $\boldsymbol{\psi}$ as follows. For each i, j and each odd t:

(ψ-i): if $\theta_{i,j,t} > 0$, then $\psi_i(x) \leq \theta_{i,j,t}$, for all $x \in B(y_{j,t}, \delta)$, with equality iff $x = y_{j,t}$;

(ψ-ii): if $\theta_{i,j,t} < 0$, then $\psi_i(x) \geq \theta_{i,j,t}$, for all $x \in B(y_{j,t}, \delta)$, with equality iff $x = y_{j,t}$;

(ψ-iii): if $\theta_{i,j,t} = 0$, then $\psi_i(x) = 0$, for all $x \in B(y_{j,t}, \delta)$;

(ψ-iv): $\psi_i(\cdot) = 0$ on the complement of $\{B(x, \delta): x \in \bigcup_{t \text{ odd}} Y_\tau\}$.

Clearly, there will exist a continuous function satisfying conditions (ψ-i) and (ψ-ii) if and only if $y_{j,t} = y_{j',t'}$ implies $\theta_{i,j,t} = \theta_{i,j',t'}$. Now by construction, we know $y_{j,t} = y_{j',t'}$ only if either $t = t'$ or $y_{j,t} = x_j^*$, for some j. By (θ-i,ii), $\theta_{i,j,t} = \theta_{i,j',t}$ in the first case; by (θ-iii), $\theta_{i,j,t} = 0$ in the second case. This verifies that there is indeed a continuous function satisfying conditions (ψ-i) and (ψ-ii).

Finally let $\mathbf{v}^n = \mathbf{u}^n + \psi$. Note that since the norm of ψ does not exceed $\bar{\varepsilon}$, $\mathbf{v}^n$ is strictly concave. Also, since $\bar{\varepsilon} < n^{-1}$, applying the construction above for every n yields a sequence $(\mathbf{v}^n)$ that converges to $\bar{\mathbf{u}}$.

We now define a strategy profile as follows. For each i, define:

(s-i): $A_{i,T} = (v_i^n)^{-1}([v_i(\varnothing), \infty])$;

(s-ii): for each even $t < T$, define $A_{i,t} = (v_i^n)^{-1}([\pi_i(Y_{t+1}, \Theta_{t+1}), \infty])$;

(s-iii): for each odd t, let $x_{i,t} = y_{i,t}$;

(s-iv): choose $C_{i,t}$ from $\mathbb{C}$ such that $y_{i,t}$ maximizes $u_i^n(\cdot)$ on $\bigcap_{j \in C_{i,t}} A_{j,t}$.

Now let $s_i^n = (x_{i,t}, C_{i,t}, A_{i,t+1})_{t=1,3,\cdots,T-1}$. We will show that $\mathbf{s} = (s_i)_{i \in I}$ is the unique equilibrium for the game $\Gamma(\mathbf{v}^n, T)$.

Proposition I states that in any equilibrium for the game $\Gamma(\mathbf{v}^n, T)$, player i must play according to (s-i) in round T. Now fix t odd and assume that in round $t+1$, players' acceptance sets are as specified either in (s-i) or (s-ii). We will argue that in round t of any equilibrium for $\Gamma(\mathbf{v}^n, T)$, player i must propose the vector $y_{i,t}$. First suppose that $y_{i,t}^* \in \text{interior}(\bigcap_{j \in C} B_{j,t})$, for some C. From (z-i), $y_{i,t} = x_i^*$ in this case; (by θ-i) $\theta_{i,i,t} = 0$ so that $u_i^n(\cdot) = v_i^n(\cdot)$ on $B(y_{i,t}, \delta)$. Hence for this C, $y_{i,t} \in \text{interior}(\bigcap_{j \in C} A_{j,t})$. Moreover, since $y_{i,t}$ globally maximizes $u_i^n(\cdot)$, it must be a local maximum on $B(y_{i,t}, \delta)$, and hence a local maximum for $v_i^n(\cdot)$. Since $v_i^n(\cdot)$ is

31 Given a set X, the symbol "∂X" denotes the boundary of X.

strictly concave, a local maximum on an open set is a global maximum, so that in this case, i must indeed specify $y_{i,t}$. Now suppose that $y_{i,t} \neq x_i^*$. In this case, we need to show that the set of *local* maximizers of $v_i^n(\cdot)$ on $\bigcup_{C \in \mathbb{C}} \bigcap_{j \in C} A_{j,t}$ is a subset of $Z_{i,t}$. It will follow immediately from this and from the definition of ψ_i that $y_{i,t}$ yields a strictly higher payoff than any other $z \in Z_{i,t}$, hence $y_{i,t}$ is player i's unique best proposal in round t.

Fix $\bar{z} \in Z_{i,t}$, $\bar{z} \neq x_i^*$. From (z-i), $\bar{z}$ is a strict local maximum on $\partial(\bigcap_{j \in C} A_{j,t})$, for for some coalition C. We claim that in this case, it is also true that

$$\bar{z} \text{ is a strict local maximum of } v_i^n(\cdot) \text{ on } \partial(\bigcap_{j \in C} A_{j,t}) \qquad (*)$$

and hence, by concavity, a global maximum on the set $\bigcap_{j \in C} A_{j,t}$. First, recall from the proof of Lemma I(i) that if $y_{j,t} = \bar{z}$, for some $j \neq i$, then $\bar{z} \notin \partial A_{j,t}$. Moreover, by (z-ii), for $\tau \neq t$. $\bar{z} \notin Y_\tau \subset Z_\tau$. These observations, together with (ψ-iv), establish that for all j such that $\bar{z} \in \partial A_{j,t}$, $v_j^n(\cdot) = u_j^n(\cdot)$ on $B(\bar{z}, \delta)$. Consequently, on $B(\bar{z}, \delta)$ the sets $\partial(\bigcap_{j \in C} A_{j,t})$ and $\partial(\bigcap_{j \in C} B_{j,t})$ coincide. Since both $\psi_i(\cdot)$ and $u_i^n(\cdot)$ are both locally maximized on $\partial(\bigcap_{j \in C} B_{j,t})$ at $\bar{z}$, $v_i^n(\cdot)$ must be locally maximized on $\partial(\bigcap_{j \in C} A_{j,t})$ at $\bar{z}$.

To complete the proof of Theorem II, fix $t \in \{2, 4, \cdots, T-2\}$, and assume that in any equilibrium for $\Gamma(\mathbf{v}^n, T)$, player i must propose $y_{i,t+1}$. The preceding paragraph established that this assumption holds for $t = T-2$. From Lemma I(ii), each proposal will be accepted in the next round. If follows from the definition of $\mathbf{v}^n(\cdot)$ that $Ev_i^n(s^n \mid t+1) = \pi_i(Y_{t+1}, \Theta_{t+1})$, so that from Proposition I, in any equilibrium for $\Gamma(\mathbf{v}^n, T)$, player i must specify the acceptance set $A_{i,t}$ in round t. $\square$

<u>Proof of Theorem IV</u>: Throughout the proof, we will assume that the space of policy vectors $X \subset \mathbf{R}$, the payoff vector, $\mathbf{u}$, and the vector of access probabilities $\mathbf{w}$ are all given. A coalition C belongs to the admissible set $\mathbb{C}$ if and only if it contains a strict majority of the players.

We begin by introducing some further notation. Define the mappings $G_i(\cdot)$ and $U_i(\cdot)$ on $\mathbf{R}$ by: $G_i(\mathbf{x}) = \{y \in \mathbf{R}: u_i(y) \geq \min_j u_i(x_j)\}$ and $U_i(\mathbf{x}) = \{y \in \mathbf{R}: u_i(y) \geq \sum_j w_j u_i(x_j)\}$. For each i and proposal profile $\mathbf{x}$, let $\underline{\beta}_i(\mathbf{x})$ and $\overline{\beta}_i(\mathbf{x})$ denote, respectively, the smallest and largest elements of $G_i(\mathbf{x})$. Finally, given a closed set $Y \subset \mathbf{R}$, let $l(Y)$ and $h(Y)$ denote, respectively, the lowest and highest elements of $\mathbf{x}$.

Moreover, the concavity of u_i implies that $u_i(E\mathbf{x}) > \sum_j w_j u_i(x_j)$, for every i, so that for every $\mathbf{x}$, $\bigcap_{i \in I} U_i(\mathbf{x})$ is nonempty. It follows immediately that $\bigcup_{C \in \mathbb{C}} \bigcap_{i \in C} U_i(\mathbf{x})$ is a convex set.

<u>Lemma IV.1</u>: For each $\varepsilon > 0$, there exists $\delta \in (0, 1)$ such that for every $C \in \mathbb{C}$ and $\mathbf{x} \subset X$ such that $h(x) - l(x) > \varepsilon$:

$$\text{if either } (a)\ \alpha_i = x_i \text{ or } (b)\ \alpha_i \notin (l(\mathbf{x}), h(\mathbf{x})), \text{ for every } i \in C, \quad \text{then}$$

$$\bigcap_{i \in C} U_i(\mathbf{x})) \in [l(\bigcap_{i \in C} G_i(\mathbf{x})) + \delta,\ h(\bigcap_{i \in C} G_i(\mathbf{x})) - \delta].$$

Proof of Lemma IV.1: If the Lemma were false, we could find $\varepsilon > 0$, a sequence of vectors, $\{x^n\}$ in X, and a sequence of coalitions C^n such that for each n, $h(\bigcap_{i\in C^n} G_i(x^n)) - l(\bigcap_{i\in C^n} G_i(x^n)) \geq \varepsilon$ and either condition (a) or (b) above is satisfied for some $i \in C^n$, while $\bigcap_{i\in C} U_i(x^n)) \not\subset [l(\bigcap_{i\in C} G_i(x^n)) + n^{-1}, \; h(\bigcap_{i\in C} G_i(x^n)) - n^{-1}]$. Pick a convergent subsequence, again indexed by n, such that for some fixed coalition C and each $i \in C$, either condition (a) is satisfied for every n or condition (b) is satisfied for every n. Let $\overline{x}$ be the limit of the subsequence. Clearly $\text{diam}(G_i(\overline{x})) \geq \varepsilon$, $h(\bigcap_{i\in C^n} G_i(\overline{x})) - l(\bigcap_{i\in C^n} G_i(\overline{x})) \geq \varepsilon$ while there exists $\overline{y} \in \bigcap_{i\in C} U_i(\overline{x})$ such that for some $i \in C$, $u_i(\overline{y}) \leq \min_j u_i(\overline{x}_j)$.

First assume that for this i, condition (b) holds for every n. We can assume without loss of generality that for every n, $\alpha_i > h(x^n)$ so that $\alpha_i \geq h(\overline{x})$. Because u_i is strictly concave, $u_i(h(\overline{x})) > u_i(l(\overline{x}))$. But in this case, $u_i(\overline{y}) \leq u_i(l(\overline{x})) < \sum_j w_j u_i(\overline{x}_j)$, contradicting the fact that $\overline{y} \in U_i(\overline{x})$. Next, assume that for this i, condition (a) holds for every n, so that $\alpha_i = \overline{x}_i$. If $u_i(h(\overline{x})) \neq u_i(l(\overline{x}))$. then we can apply the argument above to obtain a contradiction. Assume therefore that $u_i(h(\overline{x})) = u_i(l(\overline{x}))$. By strict concavity, $u_i(l(\overline{x})) < u_i(y)$, for each $y \in (l(\overline{x}), h(\overline{x}))$. Moreover, by assumption $\alpha_i = \overline{x}_i \in (l(\overline{x}), h(\overline{x}))$. Therefore, once again, $u_i(\overline{y}) \leq u_i(l(\overline{x})) < \sum_j w_j u_i(\overline{x}_j)$, contradicting the fact that $\overline{y} \in U_i(\overline{x})$. □

Fix a particular equilibrium profile s, and for $t \in \{1, 3, \cdots, T-1\}$, let $\mathbf{x}_t$ denote the profile of policy vectors proposed in round t. Note that from Theorem II, player i's acceptance set in round $t \in \{2, 4, \cdots, T-2\}$ must be $U_i(\mathbf{x}_{t+1})$. Thus, in round $t \in \{1, 3, \cdots, T-3\}$, the set of policy vectors that will be acceptable to some coalition in round t is given by $\bigcup_{C\in\mathbf{C}} \bigcap_{i\in C} U_i(\mathbf{x}_{t+2})$. Since this set is convex, it follows that if $\alpha_i \in (l(\mathbf{x}_{t+2}), h(\mathbf{x}_{t+2}))$, for some i, then if i proposes α_i, it will be accepted by some coalition. We have established, then, that for each i,

$$\text{if } \alpha_i \in (l(\mathbf{x}_{t+2}), h(\mathbf{x}_{t+2})) \quad \text{then} \quad x_{i,t+2} = \alpha_i .$$

so that the hypothesis of Lemma IV.1 is satisfied.

Let $\underline{\iota}(t, \cdot)$ and $\overline{\iota}(t, \cdot)$ be alternative enumerations of I such that for $1 \leq k < \overline{i}$, $\underline{\beta}_{\underline{\iota}(t,k)}(\mathbf{x}_t) \leq \underline{\beta}_{\underline{\iota}(t,k+1)}(\mathbf{x}_t)$, while $\overline{\beta}_{\overline{\iota}(t,k)}(\mathbf{x}_t) \geq \overline{\beta}_{\overline{\iota}(t,k+1)}(\mathbf{x}_t)$. Next, define $\tilde{i}$ to be the smallest integer strictly larger than $\overline{i}/2$ and define $\underline{I}_t = \{\underline{\iota}(t,1)), \cdots, \underline{\iota}(t,\tilde{i})\}$ and $\overline{I}_t = \{\overline{\iota}(t,1)), \cdots, \overline{\iota}(t,\tilde{i})\}$. Observe that for each t, $\underline{I}_t \cap \overline{I}_t \neq \emptyset$ and for each $\tau \neq t$, $\underline{I}_t \cap \overline{I}_\tau \neq \emptyset$. Set $\underline{\beta}_t = \underline{\beta}_{\underline{\iota}(t,\tilde{i})}(\mathbf{x}_t)$ and $\overline{\beta}_t = \overline{\beta}_{\overline{\iota}(t,\tilde{i})}(\mathbf{x}_t)$. Thus, a policy vector y is contained in $[\underline{\beta}_t, \overline{\beta}_t]$ if and only if for a strict majority of the players in I, y is weakly preferred to the *least* preferred element of $\mathbf{x}_t$. Specifically, every $y \in [\underline{\beta}_t, E\mathbf{x}_t]$, is weakly preferred to $h(\mathbf{x}_t)$ by every $i \in \underline{I}_t$, while every $y \in [E\mathbf{x}_t, \overline{\beta}_t]$, is weakly preferred to $l(\mathbf{x}_t)$ by every $i \in \overline{I}_t$.

We are now ready to proceed with the proof of the theorem. For $t \in \{1, 3, \cdots, T-3\}$ and any admissible coalition C, it is clearly the case that

$$\mathbf{x}_t \subset \bigcup_{C\in\mathbf{C}} \bigcap_{i\in C} U_i(\mathbf{x}_{t+2}) \subset \bigcup_{C\in\mathbf{C}} \bigcap_{i\in C} G_i(\mathbf{x}_{t+2}) \subset [\underline{\beta}_{t+2}, \overline{\beta}_{t+2}]. \qquad \text{(A.IV.1)}$$

The first and second inclusions are obvious. The third follows from the fact that each admissible coalition contains exactly $\tilde{i}$ members.

Now, fix $\varepsilon > 0$ and choose $\delta > 0$ for which the conclusion of Lemma IV.1 applies. We will show that if $\bar{T}$ is sufficiently large, then for $T > \bar{T}$, the solution for the T-round game will be contained in an interval of length no greater than ε. Specifically, we have shown (A.IV.1) that for each t, $\mathbf{x}_t \subset [\underline{\beta}_{t+2}, \bar{\beta}_{t+2}]$. We will show that when $h(\mathbf{x}_{t+2}) - l(\mathbf{x}_{t+2})$ exceeds ε, the interval $[\underline{\beta}_t, \bar{\beta}_t]$ will be contained in $[\underline{\beta}_{t+2}, \bar{\beta}_{t+2}]$, but its length will be shorter by at least δ. This fact will establish the theorem.

It follows from Lemma IV.1 and A.IV.1 that for each $t \in \{1, 3, \cdots, T-3\}$

$$\text{if } h(\mathbf{x}_{t+2}) - l(\mathbf{x}_{t+2}) > \varepsilon, \text{ then } \mathbf{x}_t \subset [\underline{\beta}_{t+2} + \delta, \bar{\beta}_{t+2} - \delta]. \tag{A.IV.2}$$

The next step in the proof is to show that for each $t \in \{1, 3, \cdots, T-1\}$,

$$\text{either } \underline{\beta}_t = l(\mathbf{x}_t) \text{ or } \bar{\beta}_t = h(\mathbf{x}_t). \tag{A.IV.3}$$

To see this, observe that for each i, $\underline{\beta}_i(\mathbf{x}_t) \le l(\mathbf{x}_t)$ while $\bar{\beta}_i(\mathbf{x}_t) \ge h(\mathbf{x}_t)$. Moreover, because payoffs are concave, at most one of these inequalities can be strict for any i. Thus if $\underline{\beta}_i(\mathbf{x}_t) \le \underline{\beta}_t < l(\mathbf{x}_t)$, for each $i \in \underline{I}_t$, then $\bar{\beta}_i(\mathbf{x}_t) = \bar{\beta}_t = h(\mathbf{x}_t)$, for $i \in \underline{I}_t \cap \bar{I}_t$. Since $\underline{I}_t \cap \bar{I}_t$ is nonempty, this establishes that (A.IV.3) is true. We will now assume (without loss of generality) that $\bar{\beta}_t = h(\mathbf{x}_t)$, and rewrite A.IV.2 as

$$\bar{\beta}_t \subset [\underline{\beta}_{t+2} + \delta, \bar{\beta}_{t+2} - \delta]. \tag{A.IV.2'}$$

To complete the proof of the theorem, we need to show that $\underline{\beta}_t \ge \underline{\beta}_{t+2}$. To see this, observe first that for each $i \in \underline{I}_t$,

$$u_i(\underline{\beta}_t) \ge u_i(\underline{\beta}_i(\mathbf{x}_t)) \ge u_i(\bar{\beta}_t) > u_i(\bar{\beta}_{t+2}). \tag{A.IV.4}$$

The second inequality holds because $\bar{\beta}_t = h(\mathbf{x}_t)$; the third because $\bar{\beta}_{t+2} > \bar{\beta}_t$. We now have two cases to consider. First assume that $\bar{\beta}_{t+2} = h(\mathbf{x}_{t+2})$. In this case, A.IV.4 implies that $\underline{\beta}_t \in G_i(\mathbf{x}_{t+2})$, for $i \in \underline{I}_{t+2}$, so that, immediately, $\underline{\beta}_t \ge \underline{\beta}_{t+2}$. Second assume that $\underline{\beta}_{t+2} \in l(\mathbf{x}_{t+2})$. In this case, $u_i(\bar{\beta}_{t+2}) \ge u_i(\bar{\beta}_i(\mathbf{x}_{t+2})) \ge u_i(\underline{\beta}_{t+2})$ for $i \in \underline{I}_{t+2}$. But from A.IV.4, $u_i(\underline{\beta}_t) > u_i(\bar{\beta}_{t+2})$, for each $i \in \underline{I}_t$, Since $\underline{I}_t \cap \underline{I}_{t+2}$ is nonempty, there exists i such that

$$u_i(\underline{\beta}_t) > u_i(\bar{\beta}_{t+2}) \ge u_i(\underline{\beta}_{t+2}).$$

Since u_i is concave and $\bar{\beta}_{t+2} > \underline{\beta}_{t+2}$, it now follows that $\underline{\beta}_t > \underline{\beta}_{t+2}$. $\square$.

Proof of Theorem V: The proof uses the following lemma repeatedly.

Lemma V.1: Fix $\varepsilon > 0$, an integer k, and a strictly positive probability vector $\mathbf{p} \in \Delta^{k-1}$. There exists $\delta > 0$ such that for each i and $\mathbf{y} = (y_\kappa)_{\kappa=1}^k \subset X$, $\operatorname{diam}(\mathbf{y}) \ge \varepsilon$ implies $u_i(\mathbf{p}\cdot\mathbf{y}) - \sum_\kappa p_\kappa u_i(y_\kappa) \ge \delta$.

Proof of Lemma V.1: If the Lemma were false, then we could find $\varepsilon > 0$, $i \in I$ and a sequence of vectors, $\{\mathbf{y}^n\}$ in X, such that for each n, $\operatorname{diam}(\mathbf{y}^n) \ge \varepsilon$ and $u_i(\mathbf{p}\cdot\mathbf{y}^n) - \sum_\kappa p_\kappa u_i(y^n_\kappa) < n^{-1}$. Since X is compact the sequence $\{\mathbf{y}^n\}$ has a convergent subsequence. Let $\bar{\mathbf{y}}$ be the limit of this subsequence. Since u_i is continuous, $u_i(\mathbf{p}\cdot\bar{\mathbf{y}}) \le \sum_\kappa p_\kappa u_i(\bar{y}_\kappa)$. Moreover, the diameter of $\bar{\mathbf{y}}$ is at least ε. However, since the vector $\mathbf{p}$ is strictly positive, $p \cdot \bar{y}$ is contained in the relative interior of the convex hull of $\bar{y}$. But this contradicts the assumption that u_i is strictly concave. □

We now proceed with the proof of the theorem. Let (x^T) denote the sequence of outcomes corresponding to a nested sequence of equilibrium strategy profiles for the T-round games. Assume that player #1 is an essential player. For each T, let $\theta^T = Eu_1(x^T)$.

<u>Step #1</u>: The sequence (θ^T) is a strictly increasing, Cauchy sequence.

<u>Proof of Step #1</u>: Fix an even integer T. Since player #1 is essential, each player's policy proposal in round #1 of the T+2-round game must yield player #1 a payoff of at least θ^T. Moreover, from Lemma I(a) player #1's own proposal yields a payoff strictly exceeding θ^T. This establishes that the sequence is strictly increasing. Because u_i is continuous and X is compact, $u_i(\cdot)$ is bounded on X. Hence the sequence is Cauchy.

<u>Step #2</u>: For all positive ε, there exists a $\bar{T}$ such that for each $T > \bar{T}$, $\text{diam}(x^T) < \varepsilon$.

<u>Proof of Step #2</u>: Suppose to the contrary that there exists a subsequence, (x^n), of (x^T) such that for each n, $\text{diam}(x^n) \geq \varepsilon$. From Lemma V.1, there exists $\delta > 0$ such that for each n, and each i $u_i(w \cdot x^n) - \sum_{@jink@} w_j u_i(x_j^n) \geq \delta$. It follows that for each n, player #1's own proposal in round #1 of the n+2-round game must yield a payoff that exceeds θ^n by at least δ. Thus, for each n, o $\theta^{n+2} \geq \theta^n + w_1\delta$. But this contradicts Step #1.

<u>Step #3</u>: The limit of any convergent subsequence of (x^T) is a singleton profile $\{\bar{y}\}$ such that $u_1(\bar{y}) = \bar{\theta} = \lim_T \theta^T$. Moreover, a convergent subsequence exists.

<u>Proof of Step #3</u>: The first statement follows immediately from Steps #1 and #2. The second follows from the fact that X is compact.

<u>Step #4</u>: If $\{y\}$ is the limit of a convergent subsequence of (x^T), then y belongs to the core of the underlying game. Moreover, there are at most $\bar{i}$ distinct limits of convergent subsequences.

<u>Proof of Step #4</u>: The first sentence follows from an argument identical to the proof of Theorem II. Assume that there are $\bar{k}$ distinct limits of convergent subsequences, $\{y^1, \cdots, y^{\bar{k}}\}$. From Step #3, $u_i(y^\kappa) = \bar{\theta}$, for each κ, so that for any $k \neq \kappa$, $\frac{1}{2}y^\kappa + \frac{1}{2}y^k$ yields player #1 a strictly higher payoff than either. Moreover, for each κ, since y^κ belongs to the core, it cannot be Pareto dominated; thus, there must exist $i(\kappa) > 1$ such that $U_{i(\kappa)}(y^\kappa) \cap U_1(y^\kappa)$ has an empty interior. Suppose that $i(\kappa) = i(k) = i$, for $\kappa \neq k$. Since u_i and u_1 are both strictly concave, then $\frac{1}{2}y^\kappa + \frac{1}{2}y^k$ must yield player i a higher payoff than *either* y^κ *or* y^k. But this means that either $U_i(y^\kappa) \cap U_1(y^\kappa)$ or $U_i(y^k) \cap U_1(y^k)$ has a nonempty interior. We have established, then, that $\kappa \neq k$ implies $i(\kappa) \neq i(k)$ and hence that $\bar{k} \leq \bar{i}$.

<u>Step #5</u>: For every $\varepsilon > 0$ there exists $\bar{T}$ such that for $T > \bar{T}$, $\text{diam}(x^{T+2} \cup x^T) < \varepsilon$.

<u>Proof of Step #5</u>: Suppose to the contrary that there exists $\varepsilon > 0$ and a subsequence $(x^{T^n})_{n=1}^{\infty}$, such that for each n, $\text{diam}(x^{T^n+2} \cup x^{T^n}) > 3\varepsilon$. From Step #2, we can pick $\bar{n}$ sufficiently large that for $T > T^{\bar{n}}$, the diameter of x^T is less than ε. Clearly, for such T, the distance between any point in the convex hull of x^T and any point in the convex hull of x^{T+2} must be at least ε. Pick $\bar{\delta} > 0$ such that the conclusion of Lemma V.1 holds for this ε, with $k = 2$, $p = (\frac{1}{2}, \frac{1}{2})$ and $\delta = 3\bar{\delta}$; Thus, for $T > T^{\bar{n}}$, we have for each player i,

$$u_i(\tfrac{1}{2}\mathbf{w}\cdot(x^T + x^{T+2})) - \tfrac{1}{2}\sum_j w_j(u_i(x_j^{T^*} + u_i(x_j^{T^*+2})) \geq 3\bar{\delta}. \qquad \text{(A.V.1)}$$

Next, using Step 1 and, once again, Step 2, pick $n > \bar{n}$ sufficiently large that for each i $\theta^{T^*+2} - \theta^{T^*} < delta2$ *and* for each $T \geq T^*$, $diam(u_i(\mathbf{x}^T)) < \bar{\delta}$. Let $\tilde{x} = \tfrac{1}{2}\mathbf{w}\cdot(x^{T^*} + x^{T^*+2})$. Now, in the first round of the (T^*+2)-round game, player #1 proposes $x_1^{T^*+2}$ to some coalition C. We claim that $\tilde{x}$ will be accepted by each player in C and is strictly preferred by player #1 to $x_1^{T^*+2}$. This contradicts the hypothesis that $x_1^{T^*+2}$ is player #1's best alternative at this point of the game, and hence establishes Step #5. For $i \in C$, we have

$$\sum_j w_j u_i(x_j^{T^*+2}) \geq u_i(x_1^{T^*+2}) - \delta \geq \sum_j w_j u_i(x_j^{T^*}) - \delta \qquad \text{(A.V.2)}$$

The first inequality follows from our choice of T^*; the second uses the condition for acceptance by i of $x_1^{T^*+2}$. Combining (A.V.1) and (A.V.2) yields $u_i(\tilde{x}) > \sum_j w_j u_i(x_j^{T^*}) + 2\delta$, for each $i \in C$. On the other hand for player #1, we have

$$u_i(x_1^{T^*+2}) \leq \sum_j w_j u_1(x_j^{T^*+2}) + \delta \equiv \theta^{T^*+2} + \delta \leq \theta^{T^*} + 2\delta \equiv \sum_j w_j u_i(x_j^{T^*}) + 2\delta \qquad \text{(A.V.3)}$$

Both inequalities follow from our choice of T^*. Combining (A.V.1) and (A.V.3) yields $u_i(\tilde{x}) > u_i(x_1^{T^*+2}) + \delta$ which establishes the claim above.

Step #6: The sequence $(\mathbf{x}^T)$ has a (unique) limit point.

Proof of Step #6: Let Y denote the intersection of $u_1^{-1}(\bar{\theta})$ and the core. From Step #4, Y is a finite set. If Y is a singleton set, then Step #6 follows immediately. Assume, therefore, that Y contains two distinct elements and choose $\varepsilon > 0$ such that any two elements of Y are separated by at least 3ε. From Step #4, we can pick $\bar{T}$ such that for every $T \geq \bar{T}$, $\mathbf{x}^T \in B(Y, \varepsilon)$. Thus, there is a unique policy $\bar{y} \in Y$ such that $\mathbf{x}^T \in B(\bar{y}, \varepsilon)$. Moreover, from Step #5, there exists $\bar{\bar{T}} > \bar{T}$ such that for every $T > \bar{\bar{T}}$, $\mathbf{x}^{T+2} \subset B(\mathbf{x}^T, \varepsilon)$. It now follows from the two previous sentences that for every $T > \bar{\bar{T}}$, $\mathbf{x}^T \in B(\bar{y}, \varepsilon)$. This establishes Step #6 and completes the proof of the Theorem. □

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