A Crop Yield Expectation Stochastic Process with Beta Distribution as Limit

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Abstract

The modeling of price risk in the theory and practice of commodity risk management has been developed far beyond that of crop yield risk. This is in large part due to the use of plausible stochastic price processes. We use the Pólya urn to identify and develop a model of the crop yield expectation stochastic process over a growing season. The process allows a role for agronomic events, such as growing degree days. The model is internally consistent in adhering to the martingale property. The limiting distribution is the beta, commonly used in yield modeling. By applying binomial tree analysis, we show how to use the framework to study hedging decisions and crop valuation.

Keywords: crop insurance, growing degree days, martingale, Pólya urn, stochastic process.

Introduction

Those in the business of modeling financial prices have much in common with those seeking to model crop yield distributions. Among common features, we focus on what is arguably the most important. This is identifying a form that is both easy to work with and largely consistent with available evidence. Few believe that the lognormal distribution provides a very close representation of asset price processes, hence the body of work on volatility smiles, alternative stochastic processes, and related topics (Hull, 2009). Thick tail and other concerns continue to motivate careful work on more correctly characterizing price stochastic processes (Gabaix *et al.*, 2003; Gabaix, 2009). Crop yield distribution choice is also a matter of much debate (Just and Weninger, 1999; Atwood, Shaik, and Watts, 2003; Norwood, Roberts, and Lusk, 2004; Ramírez and McDonald, 2006; Hennessy, 2009; Harri *et al.*, 2009).

Notwithstanding the aforementioned concerns about asset price models, the approach and findings in that research field have proven to be extraordinarily useful for risk managers when pricing derivatives, making investment decisions, and hedging the consequences of those decisions. This success is due primarily to the insights and mechanical approaches enabled by working with a specific stochastic price process, even if the process is not quite right. Managers will develop rules of thumb to 'fix' perceived problems, as has been the case with Black-Scholes and related models.

One intention of the present article is to develop a plausible expected yield stochastic process as a crop matures between planting and harvest. The other intention is to demonstrate the model's potential utility. When doing so one should be mindful of some consistency and other properties that are desirable, if not essential, for the process to possess. We will require that the process supports a yield distribution with bounded (non-infinite) positive realizations.

And we will require internal consistency in the sense of the martingale property.[1] Let crop prospects be assessed at several points, say, daily, during the growing season. By martingale property we mean that the present time point expectation of tomorrow's expectation of harvest yield should equal the present time expectation of harvest yield.

Another desirable property, at least as far as we are concerned, is that when expected yield is extreme (high or low) then today's belief about the variability of tomorrow's expected yield should be very low. In other words, when comparing a crop that looks like toast with one that has moderate prospects, then the expectation for the poor-looking crop should be the more resilient to new information. And the same should be true for a promising crop when compared with one having moderate prospects. The crop with the more benign history should have the resources available, either stored internally or in the soil it has access to, to overcome a temporary weather setback (Bloom, Chapin, and Mooney, 1985).

One further desirable property is that expectations should harden as harvest approaches in the sense that the variance of next-period expectations should decline as harvest nears. This is especially so when early conditions have been extreme, be they good or bad. In late August there remains little opportunity for a string of good weather days to bail out a crop that has been dealt bad weather events earlier on. Similarly, if good early-season weather has accelerated maturity then the crop may well be harvested early and so is not vulnerable to unseasonably cool days near the normal harvest date.

The expected yield stochastic process we develop is based on the Pólya and Eggenberger Urn model (Mahmoud, 2009). Variants on the model have had diverse applications, including the study of genetic mutations and characterizing the transmission of sexual disease. The model

[1] See Chapter 4 in Durrett (1996) for extensive detail on martingales.

we will use supports a stochastic process that converges to the beta distribution (p. 53 in Mahmoud). The beta is a popular stochastic yield model. It was first used by Nelson and Preckel (1989) and has been widely adopted since then (Borges and Thurman, 1994; Babcock and Hennessy, 1996; Goodwin, 2009).

Others have studied intra-season crop dynamics. Antle (1983), Antle and Hatchett (1986), Antle, Capalbo and Crissman (1994), and Saphores (2000) have appropriately modeled production decisions within a growing year. They have sought to accommodate use of information that becomes available within the crop year, for example, on intensity of the pest population. Although our model might be adapted to include production decisions, our present concern does not lie with how information and production interact. It lies with establishing an acceptable characterization of how expected yield evolves over the growing season.

An alternative approach to doing so, as in Marcus and Modest (1984), posits that expected yield follows geometric Brownian motion over the growing season. This approach is convenient in that it allows for use of the Black and Scholes machinery for valuation, and so expected value maximizing production choices. Unfortunately, the lognormal distribution must have positive skewness while it also has a heavy right tail. Neither property is associated with received knowledge on yield distributions, especially in the prime crop-growing regions of the United States (Harri *et al.*, 2009).

After presenting and explaining the model, we will demonstrate some of its properties. We will show how the model can be viewed as a binomial tree. When weather is assumed to be the origin of uncertainty, binomial tree analysis allows for dynamic hedging against a weather derivative contract. We follow this route, allowing for deliberations on crop valuation and hedging yield risk. The paper concludes with some observations on extensions and other uses of the methodology.

Pólya's Urn

We model the growing year as having $T+1$ periods at which new information becomes available. These periods commence at time $t=0$, the planting date, and continue through to $t=T$, the harvest date. We are interested in yield expectations at each time point $t \in \Omega_T \equiv \{0,1, \ldots, T\}$. With Y_T as actual harvest yield, let Ω_t be the information set available at time t. Expected harvest yield given information available at that time is written as $\mu_t = \mathbb{E}[Y_T \mid \Omega_t]$. Without loss of generality, we assume that the yield distribution has support only on the interval $[0,1]$. With y_T as true yield, appropriate choices of a and b for linear location-and-scale transformation $Y_T = (y_T - a)/b$ will ensure that $Y_T \in [0,1]$.

We also assume that the expectation has logistic form, a specification that is widely used to model plant production processes (Carrasco-Tauber and Moffitt, 1992; Tschirhart, 2000). Specifically, let

(1) $$\mu_0 = \frac{g(x)}{f(x)+g(x)};$$

where $x = (x_1, \ldots, x_N)$ is a vector of input choices defined on the positive reals while $f(x)$ and $g(x)$ are functions from the positive reals to the positive reals. Notice at this point that these functions are defined only up to a scalar constant, i.e., if $f(x)$ and $g(x)$ work then so also do $\lambda f(x)$ and $\lambda g(x)$ for any scalar λ.

As distinct from Antle (1983), we assume that production choices are made only at planting time, $t = 0$. Writing $r(x) = g(x) / f(x)$, the production function is increasing in each input choice x_i if and only if $dr(x)/dx_i \geq 0$ while it is concave in any given argument whenever $d^2r(x)/dx_i^2 \leq 0$. Writing the optimal input choices $x = x^*$ as given, we abbreviate $f^* = f(x^*)$ and $g^* = g(x^*)$.

The model is one of information-conditioned updating of yield expectations as relevant events and the yield consequences are processed. At $t = 1$, new information arrives and expected yield evolves as follows (p. 241 in Durrett, 1996; p. 53 in Mahmoud, 2009):

$$\text{(2)} \qquad \mu_1 = \begin{cases} \mu_1^+ \equiv \dfrac{g^* + c}{f^* + g^* + c} & \text{with probability } \mu_0; \\ \mu_1^- \equiv \dfrac{g^*}{f^* + g^* + c} & \text{with probability } 1 - \mu_0; \end{cases}$$

for $c > 0$. Here $c > 0$ recognizes good weather over the first growing period. It might be viewed as the benefit from good weather.

Iterate the algorithm in (2) over $t = 2$ and further to identify the general expression

$$\text{(3)} \qquad \begin{aligned} &\mu_t = \begin{cases} \mu_t^+ \equiv \dfrac{\mu_{t-1} + m(f^*, g^*, t, c)}{1 + m(f^*, g^*, t, c)} & \text{with probability } \mu_{t-1}; \\ \mu_t^- \equiv \dfrac{\mu_{t-1}}{1 + m(f^*, g^*, t, c)} & \text{with probability } 1 - \mu_{t-1}; \end{cases} \\ &m(f^*, g^*, t, c) = \frac{c}{f^* + g^* + (t-1)c}. \end{aligned}$$

This is the expected yield stochastic process that we posit over $t \in \Omega_T$. Notice that $1 > \mu_t^+ > \mu_{t-1}$ $> \mu_t^- > 0$. Also, the fact that the favorable and unfavorable outcomes depend on the prior probability, μ_{t-1}, is necessary to ensure that the process has bounded support. Bounded support is an appropriate assumption for crop yield distributions. By contrast the stochastic processes most commonly encountered in economics, Brownian motion and geometric Brownian motion, do not even have bounded unconditional variance.[2]

The change in the random variable's value is state dependent, also unlike the stochastic processes most commonly encountered in economics. We believe state dependence to be reasonable because the change in expected yield should depend on events over the whole growing season. For example, rainfall early in the season provides a buffer against drought later on. Perhaps surprisingly, the process for μ_t possesses the Markov property. This requires that the distribution of μ_t given time $t-1$ information can be conditioned on μ_{t-1} alone among historical process realizations $\{\mu_0, \mu_1, \dots, \mu_{t-1}\}$.[3] Viewing (3), the Markov property applies in an inhomogeneous manner because the distribution of μ_t at time $t-1$ depends on the value of $t-1$ as well as on the value of μ_{t-1}. The martingale property also applies.

Martingale Property

Notice from (2) that

(4) $$\mathbb{E}[\mu_1 \,|\, \Omega_0] = \mathbb{E}\big[\mathbb{E}[Y_T \,|\, \Omega_1] \,|\, \Omega_0\big] = \mu_1^+ \mu_0 + \mu_1^- (1-\mu_0) = \mu_0.$$

In general form, (3) allows us to establish that

(5) $$\mathbb{E}[\mu_t \,|\, \Omega_{t-1}] = \mathbb{E}\big[\mathbb{E}[Y_T \,|\, \Omega_t] \,|\, \Omega_{t-1}\big] = \mu_t^+ \mu_{t-1} + \mu_t^- (1-\mu_{t-1}) = \mu_{t-1}.$$

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[2] The Ornstein-Uhlenbeck process and related mean-reverting processes are commonly used to draw commodity prices back from implausibly extreme values (p. 751 in Hull, 2009). These processes do not have bounded support, and this may be appropriate to account for commodity price spike events. But they generally do have finite unconditional variance, which is appealing in the commodity context.

[3] On the Markov property see pp. 109-117, and especially pp. 116-117, in Bhattacharya and Waymire (1990) or Chapter 5 in Durrett (1996).

Thus, μ_t satisfies the martingale property that the time $t-i$ expected value of the time t expectation of yield equals the time $t-i$ expectation of yield for all $i \in \{0,1, \ldots ,t\}$.

Binomial Tree Attribute

It is useful to express the value of μ_t two steps back where we consider μ_2 to illustrate. Viewing the distribution of μ_2 in (3), substitute for μ_1 to confirm

$$\text{(6)} \qquad \mu_2 = \begin{cases} \dfrac{g^*}{f^*+g^*+2c} & \text{with probability} \quad \dfrac{f^*(f^*+c)}{(f^*+g^*+c)(f^*+g^*)}; \\ \dfrac{g^*+c}{f^*+g^*+2c} & \text{with probability} \quad \dfrac{2f^*g^*}{(f^*+g^*+c)(f^*+g^*)}; \\ \dfrac{g^*+2c}{f^*+g^*+2c} & \text{with probability} \quad \dfrac{(g^*+c)g^*}{(f^*+g^*+c)(f^*+g^*)}. \end{cases}$$

The distribution can be viewed as a two-period binomial tree, as used in derivative analysis (Hull, 2009). Figure 1 depicts this tree, where outcomes are right-most and probabilities over each of the two time intervals are given under the appropriate arrow. In general, if we let the process proceed T steps then there are $T+1$ distinct outcomes and all are in the interval $[0,1]$. In general, the realizations are given by $n(f^*, g^*, t, i, c) = (g^*+ic)/(f^*+g^*+tc)$ on $i \in \{0,1, \ldots ,t\}$. The values are evenly spaced where the least value decreases and the greatest value increases as time t increases. The associated probabilities are more involved in that they are products of terms such as $n(f^*, g^*, t-1, i, c)$ and its complement $1-n(f^*, g^*, t-1, i, c)$.

Due to the Markov property, the good-then-bad weather outcome in (6) is the same as the bad-then-good outcome. This exchangeability attribute (p. 116 in Bhattacharya and Waymire, 1990) might be challenged on agronomic grounds, but it does simplify the stochastic structure.

The property could be relaxed but at loss of tractability.[4] We are interested in the binomial tree construction so as to be able to look at dynamic hedging possibilities, and also to bring some applied and theoretical machinery from price risk modeling to bear on yield risk analysis.

Variance Attributes

Using (3) we intend to calculate the variance of yield expectations conditional on information available at the preceding time point. Write $V_{t-1}(\mu_t) = \mathbb{E}[(\mu_t - \mu_{t-1})^2 \,|\, \Omega_{t-1}]$ so that

$$(7) \qquad V_{t-1}(\mu_t) = \frac{c^2(1-\mu_{t-1})\mu_{t-1}}{(f^* + g^* + tc)^2},$$

where calculations are provided in the appendix. This equation shows that variance is maximized when predicted yield is at the midpoint of the support, $\mu_{t-1} = 0.5$. It is minimized at support extremes, $\mu_{t-1} \in \{0,1\}$. For example, if the growing season started off badly then expected yield is unlikely to move dramatically upward in light of one period's good weather. And if the growing season started well then one period's bad weather is unlikely to move expected yield markedly downward either. These observations we refer to as *resilience*, or to quote Aristotle, "One swallow does not make a summer, neither does one fine day; … ."

The sensitivity of $V_{t-1}(\mu_t)$ with respect to time is involved. Time t appears in the denominator of expression (7) and we refer to this occurrence as the deterministic effect. Time also appears as a subscript on μ_{t-1} and we refer to this as the stochastic effect. The deterministic effect acts to decrease step-ahead conditional variance as time passes. We refer to this effect as

[4] As a practical matter when hedging or assessing value, the process could be modified such that exchangeability does not apply. This is often the case when modeling financial prices with binomial trees, perhaps in order to account for dividends. It is just a matter of introducing an additional line in programming code, where the assumed distribution can be established through

hardening of the step-ahead conditional variance near harvest. The stochastic effect of time is less clear-cut in its consequence. If time takes the process toward 0 or 1 then variance will decline due to resilience. If time takes the process toward the middle then variance will increase, as the expected marginal benefit of beneficial weather is large under those circumstances.

Marginal Weather Effects

One further comment on (2)-(3) is that the spread in expectations due to good weather is increasing and concave in c. To see this, differentiate the difference in branch outcomes:

(8) $$\mu_t^+ - \mu_t^- = \frac{c}{f^* + g^* + tc}; \qquad \frac{d(\mu_t^+ - \mu_t^-)}{dc} > 0; \qquad \frac{d^2(\mu_t^+ - \mu_t^-)}{dc^2} < 0.$$

This conveys that good weather is indeed good news but there are diminishing returns to good weather. Notice too that $d(\mu_t^+ - \mu_t^-)/dt < 0$, i.e., the spread in realizations decreases as time passes. Also, the probability attached to the upper outcome increases with the value of c so that the benefits from good weather are two-fold under this stochastic process.

Beta Limiting Distribution

For price analysis with binomial trees, constant proportional up and down movements allow the tree to approximate the lognormal distribution as step size decreases and the number of steps increase such that variance is held fixed. What happens to the discrete distribution in (3) under this limit operation? Pólya (1931) demonstrated that a limiting distribution for Y_T exists, and it has density[5]

Monte Carlo simulations.

[5] See p. 53 in Mahmoud (2009) for a proof in English.

(9) $$\frac{\Gamma\left((f^* + g^*)/c\right)}{\Gamma\left(f^*/c\right)\Gamma\left(g^*/c\right)}\left(1-Y_T\right)^{\frac{g^*}{c}-1} Y_T^{\frac{f^*}{c}-1},$$

where $\Gamma(\cdot)$ is the Gamma distribution. This means that as the number of time periods to harvest, T, increases, the realizations and associated probabilities for the discrete distribution given in (3) converge on the beta distribution.

Using standard formulae for beta distribution moments, the planting time information-conditioned mean and variance of this limiting distribution are (Bain and Engelhardt, 1992)

(10) $$\begin{aligned} \mu_0 &= \mathbb{E}[Y_T \mid \Omega_0] = \frac{(g^*/c)}{(f^*/c)+(g^*/c)} = \frac{g^*}{f^*+g^*}; \\ V_0(Y_T) &= \mathbb{E}[(Y_T - \mu_0)^2 \mid \Omega_0] = \frac{cf^*g^*}{(f^*+g^*+c)(f^*+g^*)^2}. \end{aligned}$$

The distribution mean is independent of weather outcome parameter c but distribution variance is not. Of course, the martingale property requires that (1) and (10) be consistent.

Not surprisingly, variance increases with parameter c. Indeed, variance is zero under $c = 0$ while $\lim_{c\to\infty} V_0(Y_T) = \mu_0(1-\mu_0)$. The limiting distribution under $c \to \infty$ is the binomial, a special case of the beta. One set of sufficient statistics for the two-moment distribution given in (9) is $\{f^*/c, g^*/c\}$ and this is the set usually associated with the standardized beta distribution. Another set of sufficient statistics is $\{g^*/(f^*+g^*), c\}$ where $g^*/(f^*+g^*)$ is yield mean and c characterizes distribution variability.

Derivative Analysis

Weather derivatives have become popular offerings on formally organized exchanges where cooling and heating degree days are generally the most liquid contracts (Morrison, 2009). Implications for crop insurance have been explored in, e.g., Vedenov and Barnett (2004), Woodard and Garcia (2008), and Xu, Odening, and Musshoff (2008). Both the stochastic process described by (3) and the limiting distribution given in (9) identify crop yield to be dependent on stochastic weather realizations, as represented by the presence of c in (6).

In what follows we assume a one-period process where yield is realized at $t = 1$. This is at no loss of generality because we seek to identify the optimal dynamic hedge at each node along the tree.[6] An Arrow-Debreu financial instrument can be purchased at time 0 for $\$q$ where the payoff at time $t = 1$ is \$1 in the event of good weather at $t = 1$. So the payoff is

$$(11) \qquad U = \begin{cases} 1 & \text{if good weather in period 1;} \\ 0 & \text{if bad weather in period 1.} \end{cases}$$

The crop price is assumed to be known as P at $t = 0$. Applying the standard modeling approach for hedging a derivative that can be represented as a binomial tree, assume the hedger takes a time $t = 0$ long position in h weather derivative instruments.[7]

From (3), the time 1 state-contingent payoffs are

$$(12) \qquad P\mu_1 + hU = \begin{cases} \dfrac{P(g^* + c)}{f^* + g^* + c} + h & \text{with probability } \dfrac{g^*}{f^* + g^*}; \\ \dfrac{Pg^*}{f^* + g^* + c} & \text{with probability } \dfrac{f^*}{f^* + g^*}. \end{cases}$$

Set derivative position $h = h^\#$ such that payoffs are equal across weather states, i.e., $h^\# = -Pc/(f^* + g^* + c)$ where it is interesting that $h^\#$ bears a logistic relation with c. Insert $h = h^\#$ back into (12) to ascertain a time $t = 1$ risk-free portfolio value of $Pg^*/(f^* + g^* + c)$.

For continuous time interest rate r, the time $t = 0$ present value of the payout is $Pg^* e^{-r}/(f^* + g^* + c)$. With time $t = 0$ value of this crop defined as W_0, it follows that the $t = 0$

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[6] It is just necessary to calculate the hedge ratio at each node, and implement it.

[7] On binomial trees, we refer the reader to Chapter 11 in Hull (2009).

portfolio value is $W_0 + h^{\#} q = P g^* e^{-r} / (f^* + g^* + c)$ so that

$$(13) \qquad W_0 = \frac{P(g^* e^{-r} + cq)}{f^* + g^* + c}.$$

This is a risk-neutral price for the crop. It can be achieved if the standard assumptions on transactions costs and market completeness apply.

Several comments are in order regarding eqn. (13). One is that if $c = 0$, so that $V_0(Y_T) = 0$ by (10), then $W_0 = P g^* e^{-r} / (f^* + g^*) = P\mu_0 e^{-r}$. This is the discounted expected crop value, as should be the case because yield is known with certainty at the beginning of the growing season while price is also assumed to be known. In that case, $h^{\#} = 0$ and good weather state price q does not enter the valuation because hedging serves no purpose. No model on how risk is priced is required when $c = 0$. In general, though, a stance on pricing risk is required.

One approach to valuation is to work with the risk-neutral measure (Hull, 2009) if it exists, is unique, and can be found. In our case the risk-neutral measure is easy to obtain. If π is the risk-neutral probability of favorable weather, according to the risk-neutral valuation approach then (2) and (13) provide

$$(14) \qquad W_0 = \frac{P(g^* e^{-r} + cq)}{f^* + g^* + c} = P e^{-r} \left[\frac{(g^* + c)}{f^* + g^* + c}\pi + \frac{g^*}{f^* + g^* + c}(1-\pi) \right] = \frac{P e^{-r}(g^* + \pi c)}{f^* + g^* + c}.$$

Upon cancellation, it follows that $\pi = q e^{r}$. Thus the risk-neutral, as distinct from true, probability of a favorable weather event over the period is taken as the forward discounted market valuation of an Arrow-Debreu derivative that pays off \$1 upon the event. Of course, risk preferences are not absent in the above; it is just that they have been corralled into a premium embedded in price q.

This brings us to formally modeling the good weather state price $\$q$ where the payoff is \$1 in the event of good weather for time 1. Let the good weather state bear a Capital Asset Pricing Model (CAPM) beta coefficient of β_q with the good weather state. Rate r_m is the overall market rate of return, where $r_m > r$ is assumed. Then the continuous time equilibrium expected rate of return on the binary payout good weather event asset will be $r_q = r + \beta_q(r_m - r)$.

Since the expected payout is μ_0, the state price will be $q = \mu_0 \exp[-r - \beta_q(r_m - r)]$. Insert this value together with $\mu_0 = g^* / (f^* + g^*)$ into (13) to obtain

$$(15) \qquad W_0 = P\mu_0 e^{-r} \frac{[f^* + g^* + ce^{-\beta_q(r_m - r)}]}{f^* + g^* + c},$$

whereas the crop value under risk neutrality is $P\mu_0 e^{-r}$. The risk premium is

$$(16) \qquad P\mu_0 e^{-r} - W_0 = P\mu_0 e^{-r} c \frac{[1 - e^{-\beta_q(r_m - r)}]}{f^* + g^* + c}.$$

If good crop growing weather is positively correlated with overall market returns, or $\beta_q > 0$, then $1 > e^{-\beta_q(r_m - r)}$ and early season crop value under CAPM is smaller than the expected payout.

Conclusion

This paper has provided a way to model expected yield as an information-conditioned stochastic process over the course of a crop growing season. The process has as its limiting distribution a commonly estimated stochastic crop production technology. The intent is to provide for yield randomness a technical machinery similar to that which is available for commodity price randomness. In the presence of appropriate financial instruments such as weather derivatives, the process allows for the identification of optimal dynamic hedging strategies. In the absence of such instruments, then risk managers exposed to yield risk can at least use the framework to sharpen their real-time risk assessments and make loss provisions accordingly.

Concerning further developments, an approach to account for price-yield correlation should be of interest to those exposed to crop revenue risk. This would include the providers of revenue insurance contracts. Methods are available to modeling bivariate discrete time stochastic processes for financial purposes, as in Boyle (1988) and Ho, Stapleton, and Subrahmanyam (1995). So far as we know, the methods have been confined to approximating multivariate geometric Brownian motion and so are not appropriate for the revenue insurance context.

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Appendix

<u>*Demonstration of eqn. (7)*</u>: Write $\mathbb{E}[(\mu_t - \mu_{t-1})^2 \mid \Omega_{t-1}]$ as

$$\left[\frac{\mu_{t-1} + m(f^*, g^*, t, c)}{1 + m(f^*, g^*, t, c)} - \mu_{t-1}\right]^2 \mu_{t-1} + \left[\frac{\mu_{t-1}}{1 + m(f^*, g^*, t, c)} - \mu_{t-1}\right]^2 (1 - \mu_{t-1})$$

(A1)
$$= \frac{1}{[1 + m(f^*, g^*, t, c)]^2} \frac{c^2 (1 - \mu_{t-1})^2 \mu_{t-1}}{[f^* + g^* + (t-1)c]^2} + \frac{1}{[1 + m(f^*, g^*, t, c)]^2} \frac{c^2 \mu_{t-1}^2 (1 - \mu_{t-1})}{[f^* + g^* + (t-1)c]^2}$$

$$= \frac{c^2 \left[(1 - \mu_{t-1})^2 \mu_{t-1} + \mu_{t-1}^2 (1 - \mu_{t-1})\right]}{[1 + m(f^*, g^*, t, c)]^2 [f^* + g^* + (t-1)c]^2} = \frac{c^2 (1 - \mu_{t-1}) \mu_{t-1}}{[f^* + g^* + tc]^2}.$$

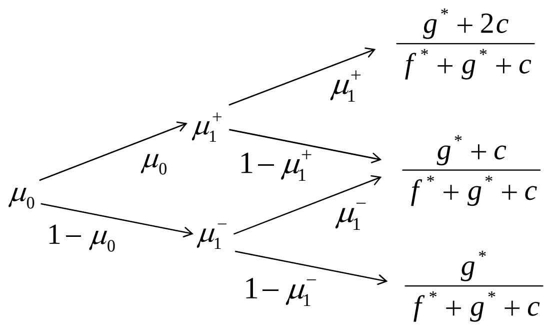

Figure 1. Binomial tree for yield stochastic process, probabilities under arrows