Option Pricing on Renewable Commodity Markets

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Abstract

The paper motivates and proposes a closed-form option-pricing model for markets such as grains or livestock where the price level can be expected to revert to expected production costs. The model suggests that traditional option pricing models will overprice long-term options on these markets.

Key words: mean reversion, option pricing, renewable commodity markets.

OPTION PRICING ON RENEWABLE COMMODITY MARKETS

Introduction

The Black and Black-Scholes option-pricing models assume that spot price volatility increases proportionally to the square root of time. This assumption is reasonable for stocks and currencies but is inconsistent with mean reversion in spot prices. Most agricultural commodity markets demonstrate a mean reversion to production costs (Bessembinder et al.), which suggests that the price volatility around this production cost reaches a maximum value. If this is true, and if price volatility is incorrectly assumed to increase in proportion to the square root of time beyond this maximum value, the fair value of long-term options will be overestimated. This problem is apparent in long-term options on crude oil futures. Schwartz recognized this problem in the context of oil futures. He had the insight that price imbalances caused by temporary shortages and surpluses would eventually disappear without affecting the long-run volatility level. For example, a shortage of oil can make the convenience yield greater than the storage cost, and this can cause nearby futures prices to exceed the prices of more distant contracts. Miltersen and Schwartz, and Hilliard and Reis proposed closed-form option-pricing models that incorporate reversion to the mean in this convenience yield. However, their models assume that the price level trends rather than reverts to a long-run mean. Therefore, such models most likely are relevant to exhaustible commodity markets such as gold and oil where Hotelling's Principle might be expected to hold.

Our interest is in renewable commodity markets such as grain or livestock, and here we can expect mean reversion in both the level of prices and the convenience yield (Routledge, Seppi, and Spatt). For example, suppose that grain prices are high because of yield shortfalls. Then, we will see a high price level across all futures contracts and an inverted market. With reversion to the mean in convenience yield only we might expect that the futures prices would eventually reflect a normal cost-of-carry market, but we

would also assume that the current high price level is permanent. With the restriction that the mean price level will revert, we also can predict that the price level will revert to the expected production cost. This additional piece of information allows us to reduce the future volatility level.

The model we propose contains Schwartz's model as a special case. Hence, it is possible to test whether the restriction imposed by Schwartz's model is warranted by observed data.

Graphical Examples

Figures 1, 2, and 3 are a graphical representation of the three assumptions that underlie the three models. All three figures show the same simulated time series of (the logarithm of) prices, and all contain the upper and lower confidence intervals for these prices at two points in time.

Figure 1 shows price under the standard Black-Scholes assumption of Brownian motion. It can be observed that the confidence interval for prices increases in proportion to the square root of time as is assumed in the model. The heavy solid line shows the expected price path and this demonstrates a small amount of growth as might be expected for the cash prices for commodities or stocks. If futures markets existed for this commodity, this heavy line would reflect the temporal basis. At time 20 in Figure 1 the cash price is lower than was expected at time 0, and the heavy dotted line shows the expected price path from this lower point. All of the price reduction from time 0 through time 20 is viewed as permanent in this model. Therefore, the updated expected price path runs parallel to the original but at a level that reflects the underperformance of price between times 0 and 20.

Figure 2 shows the Schwartz model, and is otherwise identical to Figure 1. A key difference between Figure 1 and 2 is that when the price path is updated at time 20, the Schwarz model recognizes that the price drop that occurred just before time 20 was in part due to a temporary reduction in the convenience yield reflecting a temporary surplus of the commodity. The model assumes that this temporary component will gradually disappear, and therefore it adjusts the expected time path of cash prices for this expected

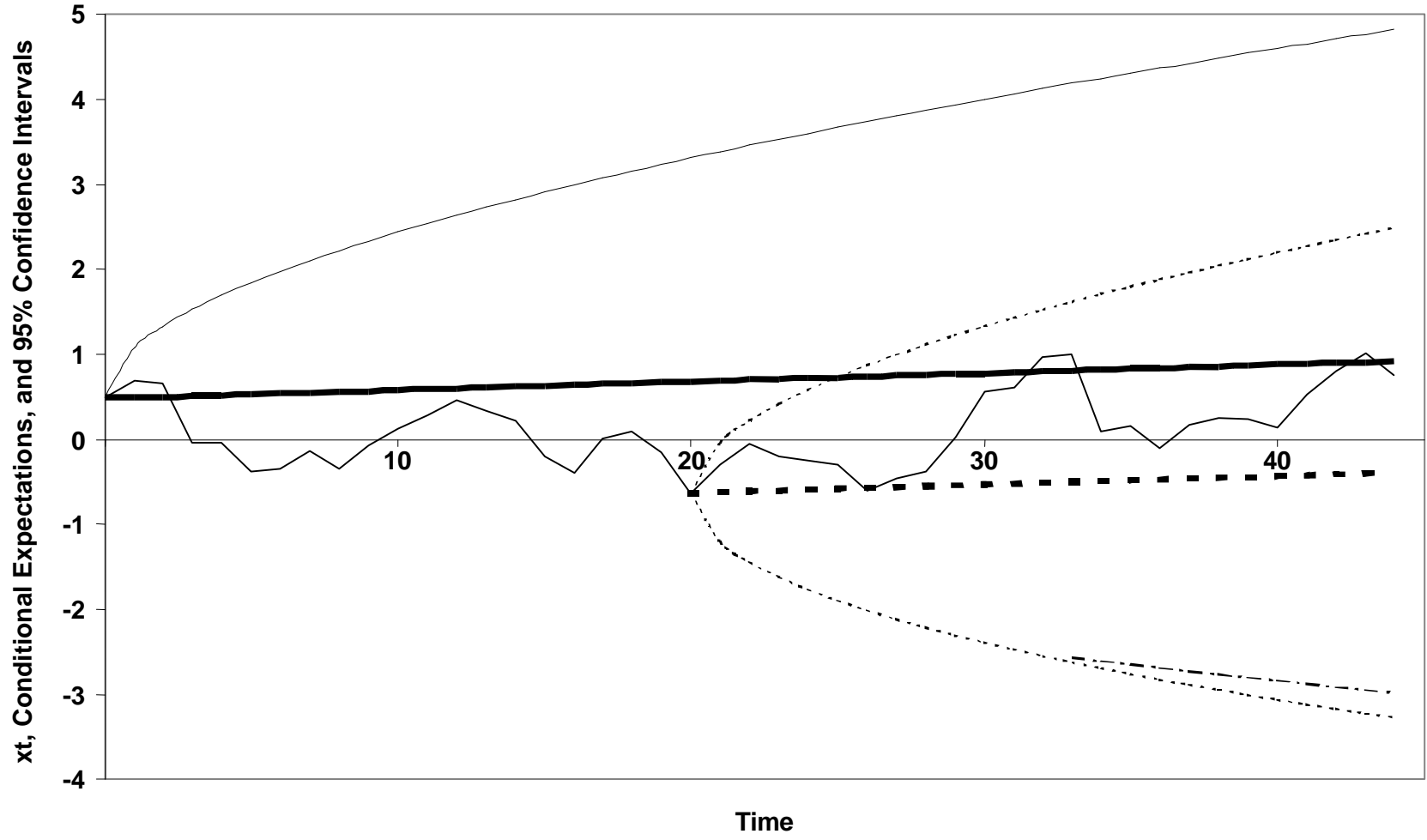**FIGURE 1. Behavior of xt, conditional expectations, and 95 percent confidence intervals under Brownian motion**
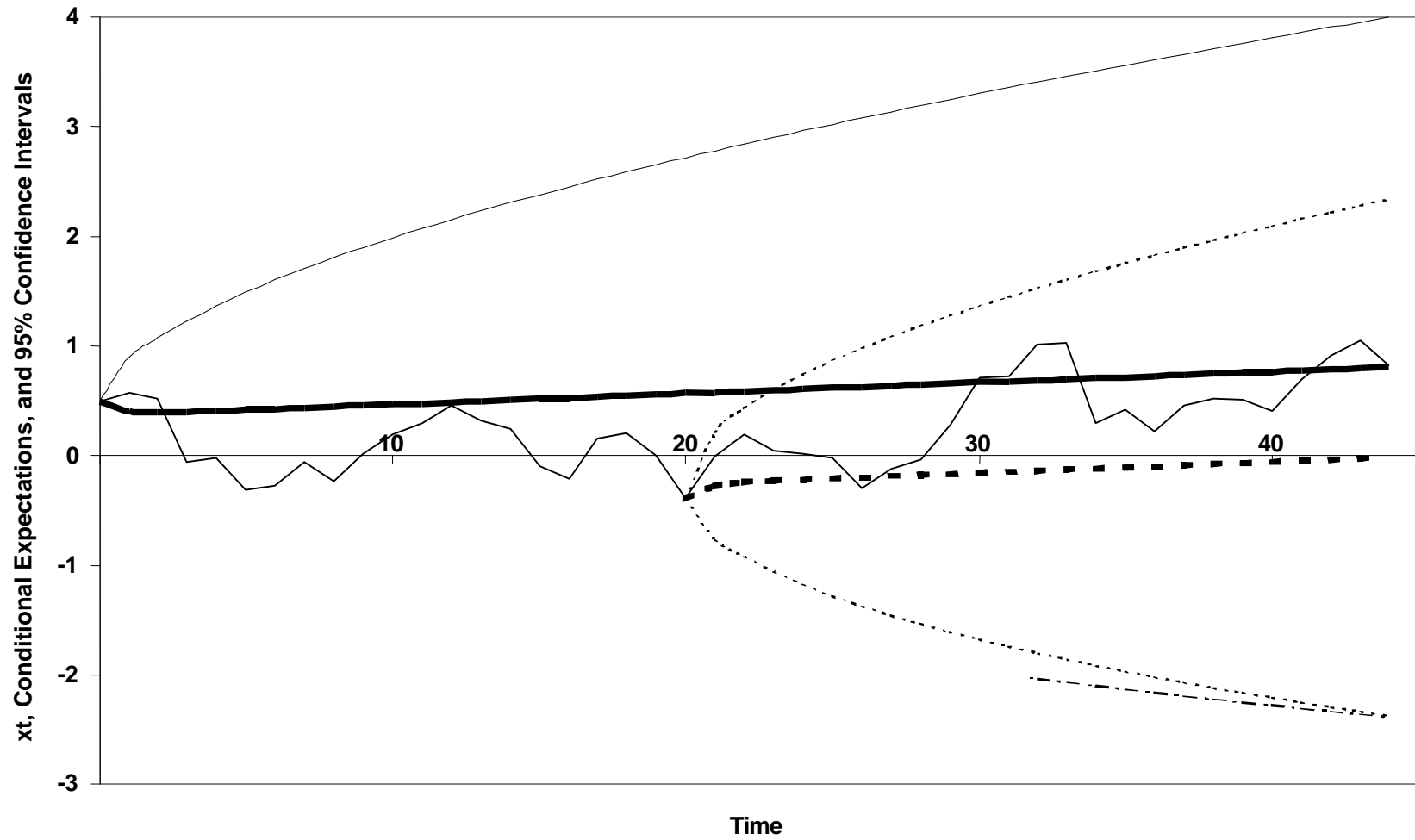
FIGURE 2. Behavior of xt, conditional expectations, and 95 percent confidence intervals under mean reversion in yt but not in xt

price recovery. However, once this temporary adjustment is out of the way, the Schwartz model behaves very much like the Black-Scholes model.

Figure 3 shows the model we propose here. The price path after time 20 contains an adjustment to the temporary imbalance as in the Schwartz model. However, the model also contains one additional piece of information. It recognizes that the generally low level of prices observed at time 20 is well below the production costs for this commodity. This suggests a reduction in supply until prices recover to these expected production costs. Therefore, the heavy dotted line approaches the heavy solid line as the model implicitly adjusts supply and demand so that expected future prices lie on the path representing expected production costs. This additional piece of information has a dramatic effect on the upper and lower confidence levels because the model recognizes that all price deviations around these expected production costs are of a temporary nature and it therefore tightens the confidence interval around this price path.

The upper and lower confidence intervals are directly related to the fair option price and we therefore have intuitive evidence suggesting that models that incorporate mean reversion in convenience yields will exhibit lower option prices than those that do not. We also can conclude that when mean reversion in the price level is added to mean reversion in the convenience yield, the fair option value will be lower still. The degree to which models that neglect mean reversion in the price level overprice option premia will of course depend on the parameters of the models, but it is clear that the degree of overpricing will increase with the time to expiration of the option.

The Schwartz Model

Schwartz advanced a path-breaking model of commodity prices that allows for mean reversion in the convenience yield but not on the spot price. Given the seminal nature of his work, and the fact that his model is a special case of the model advocated here, we introduce Schwartz's framework first.

Schwartz's fundamental insight was that commodities are characterized by "convenience yields" that are stochastic and mean reverting. Accordingly, he postulated that the convenience yield net of storage costs, $c_t \equiv$ Convenience Yield – Storage Cost, follows the Ornstein-Uhlenbeck stochastic process (1.1):
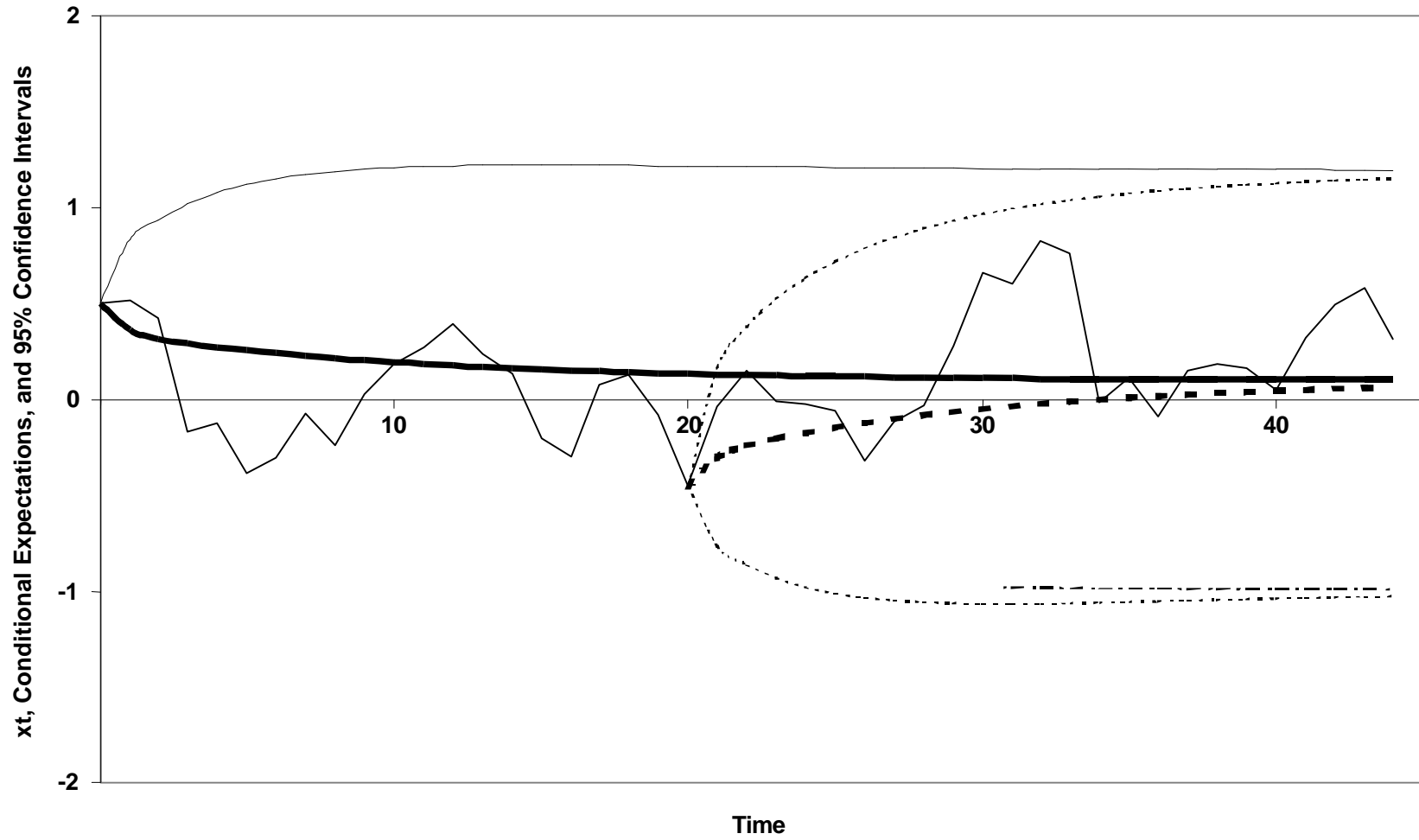
FIGURE 3. Behavior of xt, conditional expectations, and 95 percent confidence intervals under mean reversion in xt and yt

$$dc = (\boldsymbol{m}_c - \boldsymbol{k}_c \, c) \, dt + \boldsymbol{s}_c \, dW_c, \tag{1.1}$$

where $\boldsymbol{m}_c/\boldsymbol{k}_c$ = long-term mean, $\boldsymbol{k}_c \equiv$ speed of mean reversion ($\boldsymbol{k}_c > 0$), and dW_c is a Wiener process. The mean of c_T as of time t is $\boldsymbol{m}_c/\boldsymbol{k}_c + \exp[-\boldsymbol{k}_c \, (T - t)] \, (c_t - \boldsymbol{m}_c/\boldsymbol{k}_c)$, and the variance is $0.5 \, \{1 - \exp[-2 \, \boldsymbol{k}_c \, (T - t)]\} \, \boldsymbol{s}_c^2/\boldsymbol{k}_c.$[1]

In contrast to the convenience yield process (1.1), Schwartz assumed that the actual process for the commodity spot price (S) is not mean reverting. More specifically, he hypothesized that spot prices behave as a geometric Brownian motion:

$$dS = \boldsymbol{m}_S \, S \, dt + \boldsymbol{s}_S \, S \, dW_S, \tag{1.2}$$

where dW_S is a Wiener process correlated with dW_c, so that $dW_S \, dW_c = \boldsymbol{r}_{Sc} \, dt$ ($\boldsymbol{r}_{Sc}$ being the correlation coefficient). Letting $x \equiv ln(S)$, application of Ito's Lemma to (1.2) yields the arithmetic Brownian motion (1.3) for the logarithm of spot prices:

$$dx = \boldsymbol{m}_x \, dt + \boldsymbol{s}_x \, dW_x, \tag{1.3}$$

where $\boldsymbol{m}_x \equiv \boldsymbol{m}_S - \boldsymbol{s}_S^2/2$, $\boldsymbol{s}_x \equiv \boldsymbol{s}_S$, $dW_x \equiv dW_S$, and $\boldsymbol{r}_{xc} = \boldsymbol{r}_{Sc}$. In (1.2), $\boldsymbol{m}_x$ denotes the drift in the logarithm of spot prices.

The rate of return to the commodity holding consists of the relative price change ($dS/S = dx$) plus the convenience yield net of storage costs (c). Thus, the expected rate of return to commodity holders is $\boldsymbol{m}_x + c$. In equilibrium, the latter must equal the risk-free rate of return (r) plus the risk premium ($\boldsymbol{l}$). Letting $\boldsymbol{m}_x + c = r + \boldsymbol{l}$ in (1.1) and (1.3) yields the corresponding risk-neutralized stochastic processes:

$$dc = (\boldsymbol{m}_c - \boldsymbol{k}_c \, c - \boldsymbol{l}_c) \, dt + \boldsymbol{s}_c \, dW_c^*, \tag{1.4}$$

$$dx = (r - c) \, dt + \boldsymbol{s}_x \, dW_x^*, \tag{1.5}$$

where $\boldsymbol{l}_c$ is the market price for c risk and dW_c^* and dW_x^* are the Wiener processes under the equivalent martingale measure. Note that $dW_x^* \, dW_y^* = \boldsymbol{r}_{xy} \, dt$. Schwartz derived futures prices under the aforementioned assumptions, whereas Miltersen and Schwartz, and Hilliard and Reis obtained the equations for the corresponding option prices.

Allowing for Mean Reversion in the Price Level

Unlike Schwartz, here spot prices are allowed to be mean reverting as in (2.1):

$$dS = [\boldsymbol{m}_S - \boldsymbol{k}_S \ ln(S)] \ S \ dt + \boldsymbol{s}_S \ S \ dW_S. \tag{2.1}$$

Given (2.1), Ito's Lemma yields the Ornstein-Uhlenbeck stochastic process (2.2) for the logarithm of the spot prices:

$$dx = (\boldsymbol{m}_x - \boldsymbol{k}_x \ x) \ dt + \boldsymbol{s}_x \ dW_x, \tag{2.2}$$

where $\boldsymbol{k}_x \equiv \boldsymbol{k}_S > 0$ is the speed at which the logarithm of the spot price reverts to its long-run mean $\boldsymbol{m}_x/\boldsymbol{k}_x$. Note that if $\boldsymbol{k}_x = 0$, (2.2) collapses to (1.3), that is, the arithmetic Brownian motion with expected drift $\boldsymbol{m}_x$ assumed by Schwartz.

A stylized fact of commodity markets is that convenience yields are positively associated with spot prices.[2] Hence, the convenience yield net of storage costs is postulated to consist of the following random function of the logarithm of the spot price:

$$c = y + \boldsymbol{k}_x \ x, \tag{2.3}$$

where y follows the Ornstein-Uhlenbeck stochastic process (2.4):

$$dy = (\boldsymbol{m}_y - \boldsymbol{k}_y \ y) \ dt + \boldsymbol{s}_y \ dW_y. \tag{2.4}$$

Wiener processes dW_x and dW_y are correlated so that $dW_x \ dW_y = \boldsymbol{r}_{xy} \ dt$, where $\boldsymbol{r}_{xy}$ is the correlation coefficient. Setting $\boldsymbol{k}_x = 0$ in (2.3) yields $c = y$, in which case the advocated convenience yield process becomes identical to that in Schwartz.

In equilibrium, the instantaneous expected total return to commodity holders $\{E(dS/S + c) = [(\boldsymbol{m}_x - \boldsymbol{k}_x \ x) + (y + \boldsymbol{k}_x \ x)] \ dt\}$ must equal the risk-free rate plus the associated market price of risk $(r + \boldsymbol{l})$. Therefore, the risk-neutral process for dx may be written as (2.5)

$$dx = [r - (y + \boldsymbol{k}_x \ x)] \ dt + \boldsymbol{s}_x \ dW_x^*, \tag{2.5}$$

where dW_x^* is the Wiener process under the equivalent martingale measure. Component y of the convenience yield (2.3) cannot be hedged because it is not traded. Hence, the

stochastic process for y under the equivalent martingale measure (2.6) depends on the market price for y risk ($\boldsymbol{l}_y$):

$$dy = (\boldsymbol{m}_y - \boldsymbol{k}_y \, y - \boldsymbol{l}_y) \, dt + \boldsymbol{s}_y \, dW_y^*. \tag{2.6}$$

In (2.6), dW_y^* is the Wiener process under the equivalent martingale measure. Note that $dW_x^* \, dW_y^* = \boldsymbol{r}_{xy} \, dt$.

The risk-neutralized processes (2.5) and (2.6) provide the foundations to derive commodity futures and options prices, which are discussed in the next two sections.

Futures Prices

Under the assumption that the risk-free interest rate r is constant, at time t the commodity futures price with maturity T is simply the time-t expected price of the commodity at time T under the equivalent martingale measure. That is,

$$F(S_t, y_t, t, T) = E_t^*(S_T), \tag{3.1}$$

where $E_t^*(\cdot)$ denotes the expectation with respect to the risk-neutralized processes (2.5) and (2.6). The expression for $E_t^*(S_T)$ can be obtained by noting that $S_T = exp[ln(S_T)] = exp(x_T)$, and that the vector (x, y) follows an affine diffusion (e.g., Dai and Singleton) under the martingale measure. This allows us to apply the method proposed by Duffie, Pan, and Singleton to get a closed-form solution for the futures price.

The expression for the futures price is (3.2):

$$F(S_t, y_t, t, T) = exp\{ \boldsymbol{A}(0) - \boldsymbol{A}(t - T) + \frac{1}{2} \, [\boldsymbol{F}(0) - \boldsymbol{F}(t - T)] \tag{3.2}$$

$$+ ln(S_t) \, \boldsymbol{B}_x(t - T) + y_t \, \boldsymbol{B}_y(t - T) \},$$

where

$$\boldsymbol{A}(t) \equiv \left[r - \frac{(\boldsymbol{m}_y - \boldsymbol{l}_y)}{\boldsymbol{k}_y} \right] \frac{\boldsymbol{B}_x(t)}{\boldsymbol{k}_x} + (\boldsymbol{m}_y - \boldsymbol{l}_y) \frac{\boldsymbol{B}_y(t)}{\boldsymbol{k}_y},$$

$$F(t) \equiv \left[s_x^2 + \frac{(s_y^2 - 2r_{xy}s_x s_y k_y)}{k_y(k_x + k_y)} \right] \frac{B_x(t)^2}{2k_x}$$

$$- \frac{(s_y^2 - 2r_{xy}s_x s_y k_y)}{k_y(k_x + k_y)} B_x(t) B_y(t) + s_y^2 \frac{B_y(t)^2}{2k_y},$$

$$B_x(t) \equiv exp(k_x\, t),$$

$$B_y(t) \equiv \frac{exp(k_x t) - exp(k_y t)}{k_x - k_y}.$$

Options on Futures Contracts

Of particular interest here are the prices of options on futures. To this end, let $C[F(S_t,$ $y_t, t, T), K, t, T_1]$ denote the price at time t of a European call option expiring at time $T_1 \geq$ t on a futures contract that expires at time $T \geq T_1$, with strike price K. Since the payoff of such an option at expiration is $max[F(S_{T_1}, y_{T_1}, T_1, T) - K, 0]$, standard arguments can be applied to show that its price at time t is given by

$$C[F(S_t, y_t, t, T), K, t, T_1] = exp[r\,(t - T_1)]\; E_t^*\{max[F(S_{T_1}, y_{T_1}, T_1, T) - K, 0]\}. \quad (4.1)$$

The analytical solution for the call option price $C[F(S_t, y_t, t, T), K, t, T_1]$ can be computed by resorting again to Duffie, Pan, and Singleton, as we do next.

The moment-generating function of the logarithm of futures prices under the equivalent martingale measure is defined by (4.2):

$$M_{ln[F(S_{T_1}, y_{T_1}, T, T)]}(z) \equiv E_t^*\{exp[z\, ln(F(S_{T_1}, y_{T_1}, T_1, T))]\}. \quad (4.2)$$

Using (3.2) to substitute for $ln[F(S_{T_1}, y_{T_1}, T_1, T)]$ on the right-hand side of (4.2) and rearranging yields the following expression for the moment-generating function:

$$M_{ln[F(S_{T_1}, y_{T_1}, T_1, T)]}(z) = exp\{z\,[A(0) - A(T_1 - T) + \frac{1}{2}\,(F(0) - F(T_1 - T))]\} \quad (4.3)$$

$$E_t^* \{ exp[z \ \boldsymbol{B}_x(T_1 - T) \ x_{T_1} + z \ \boldsymbol{B}_y(T_1 - T) \ y_{T_1}] \}.$$

The expectation term on the right-hand side of (4.3) is of the same form as equation (2.3) in Duffie, Pan, and Singleton, so these authors' advocated method can be applied. The resulting analytical solution for the moment-generating function is (4.4):

$$M_{ln[F(S_{T_1}, y_{T_1}, T_1, T)]}(z) = exp[\boldsymbol{m}(S_t, y_t, t, T_1, T) \ z + \frac{1}{2} \ \boldsymbol{s}(t, T_1, T)^2 \ z^2], \qquad (4.4)$$

where

$$\boldsymbol{m}(S_t, y_t, t, T_1, T) \equiv \boldsymbol{A}(0) - \boldsymbol{A}(t - T) + \frac{1}{2} \ [\boldsymbol{F}(0) - \boldsymbol{F}(T_1 - T)]$$

$$+ \ ln(S_t) \ \boldsymbol{B}_x(t - T) + y_t \ \boldsymbol{B}_y(t - T),$$

$$\boldsymbol{s}(t, T_1, T)^2 \equiv \boldsymbol{F}(T_1 - T) - \boldsymbol{F}(t - T).$$

The specific form of moment-generating function (4.4) implies that $ln[F(S_{T_1}, y_{T_1}, T_1, T)]$ is distributed as a normal random variable with mean $\boldsymbol{m}(S_t, y_t, t, T_1, T)$ and variance $\boldsymbol{s}(t, T_1, T)^2$. In addition, (4.4) implies that $F(S_t, y_t, t, T) = exp[\boldsymbol{m}(S_t, y_t, t, T_1, T) + \boldsymbol{s}(t, T_1, T)^2/2]$. This is true because $F(S_t, y_t, t, T) = E_t^*[F(S_{T_1}, y_{T_1}, T_1, T)]$, and the latter expectation equals $M_{ln[F(S_{T_1}, y_{T_1}, T_1, T)]}(z = 1)$ (see [4.2]). These two results make it straightforward to derive the following analytical solution for the price of the call option:

$$C[F(S_t, y_t, t, T), K, t, T_1] = exp[r \ (t - T_1)] \ [F(S_t, y_t, t, T) \ N(d_1) - K \ N(d_2)], \qquad (4.5)$$

where $N(\cdot)$ is the standard normal cumulative probability distribution, $d_1 \equiv \{ln[F(S_t, y_t, t, T)/K] + 0.5 \ \boldsymbol{s}(t, T_1, T)^2\}/\boldsymbol{s}(t, T_1, T)$, and $d_2 \equiv \{ln[F(S_t, y_t, t, T)/K] - 0.5 \ \boldsymbol{s}(t, T_1, T)^2\}/\boldsymbol{s}(t, T_1, T)$.

Endnotes

1. The discrete-time analog of process (1.1) is $c_t = \phi_0 + \phi_1 c_{t-1} + $ i.i.d. *shock*$_t$.

2. Typically, when a commodity is in relatively short supply, its price is high and its convenience is high as well.

Appendix A

The affine diffusion for vector (x, y) under the risk-neutral measure may be written as follows:

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \left(\begin{bmatrix} r \\ \boldsymbol{m}_y - \boldsymbol{1}_y \end{bmatrix} - \begin{bmatrix} \boldsymbol{k}_x & 1 \\ 0 & \boldsymbol{k}_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) dt + \begin{bmatrix} \boldsymbol{s}_x & 0 \\ \boldsymbol{r}_{xy}\boldsymbol{s}_y & \sqrt{(1 - \boldsymbol{r}_{xy}^2)}\boldsymbol{s}_y \end{bmatrix} \begin{bmatrix} dW_x^* \\ dW_y^* \end{bmatrix}. \quad \text{(A.1)}$$

Appendix B

The call option formula can be obtained by noting that if $ln(S)$ is normally distributed with mean m and variance s^2, then

$$E[max(F_{T_1} - K), 0] = \int_{ln(K)}^{\infty} (F_{T_1} - K)\frac{1}{s\sqrt{2p}} exp\left\{-\frac{1}{2}\left[\frac{ln(F_{T_1}) - m}{s}\right]^2\right\} d[ln(F_{T_1})] \quad \text{(B.1)}$$

$$= \int_{ln(K)}^{\infty} F_{T_1}\frac{1}{s\sqrt{2p}} exp\left\{-\frac{1}{2}\left[\frac{ln(F_{T_1}) - m}{s}\right]^2\right\} d[ln(F_{T_1})]$$

$$- K\int_{ln(K)}^{\infty}\frac{1}{s\sqrt{2p}} exp\left\{-\frac{1}{2}\left[\frac{ln(F_{T_1}) - m}{s}\right]^2\right\} d[ln(F_{T_1})],$$

because $F_{T_1} > K \Rightarrow ln(F_{T_1}) > ln(K)$. But $ln(F_{T_1}) > ln(K) \Rightarrow [ln(F_{T_1}) - m]/s > [ln(K) - m]/s = [ln(K) - m - s^2/2 + s^2/2]/s = [ln(K) - ln(F_t) + s^2/2]/s$, so that

$$\int_{ln(K)}^{\infty}\frac{1}{s\sqrt{2p}} exp\left\{-\frac{1}{2}\left[\frac{ln(F_{T_1}) - m}{s}\right]^2\right\} d[ln(F_{T_1})] = 1 - N\{[ln(K/F_t) + s^2/2]/s\} \quad \text{(B.2)}$$

$$= N\{[ln(F_t/K) - s^2/2]/s\} \quad \text{(B.2')}$$

In addition,

$$\int_{ln(K)}^{\infty} F_{T_1}\frac{1}{s\sqrt{2p}} exp\left\{-\frac{1}{2}\left[\frac{ln(F_{T_1}) - m}{s}\right]^2\right\} d[ln(F_{T_1})] \quad \text{(B.3)}$$

$$= \int_{ln(K)}^{\infty}\frac{1}{s\sqrt{2p}} exp\left\{ln(F_{T_1}) - \frac{1}{2}\left[\frac{ln(F_{T_1}) - m}{s}\right]^2\right\} d[ln(F_{T_1})]$$

$$= \int_{ln(K)}^{\infty} \frac{1}{\boldsymbol{s}\sqrt{2\boldsymbol{p}}} \exp\left\{ \boldsymbol{m} + \frac{\boldsymbol{s}^2}{2} - \frac{1}{2}\left[\frac{ln(F_{T_1}) - (\boldsymbol{m} + \boldsymbol{s}^2)}{\boldsymbol{s}} \right]^2 \right\} d[ln(F_{T_1})]$$

$$= \exp(\boldsymbol{m} + \boldsymbol{s}^2/2) \int_{ln(K)}^{\infty} \frac{1}{\boldsymbol{s}\sqrt{2\boldsymbol{p}}} \exp\left\{ -\frac{1}{2}\left[\frac{ln(F_{T_1}) - (\boldsymbol{m} + \boldsymbol{s}^2/2) - \boldsymbol{s}^2/2}{\boldsymbol{s}} \right]^2 \right\} d[ln(F_{T_1})]$$

$$= F_t \int_{ln(K)}^{\infty} \frac{1}{\boldsymbol{s}\sqrt{2\boldsymbol{p}}} \exp\left\{ -\frac{1}{2}\left[\frac{ln(F_{T_1}) - ln(F_t) - \boldsymbol{s}^2/2}{\boldsymbol{s}} \right]^2 \right\} d[ln(F_{T_1})]$$

$$= F_t \{1 - N[(ln(K/F_t) - \boldsymbol{s}^2/2)/\boldsymbol{s}]\}$$

$$= F_t \, N\{[ln(F_t/K) + \boldsymbol{s}^2/2]/\boldsymbol{s}\} \tag{B.3'}$$

Substituting (B.2') and (B.3') into (B.1) yields

$$E[max(F_{T_1} - K), 0] = F_t \, N\{[ln(F_t/K) + \boldsymbol{s}^2/2]/\boldsymbol{s}\} - K \, N\{[ln(F_t/K) - \boldsymbol{s}^2/2]/\boldsymbol{s}\}. \tag{B.4}$$

The call option formula (4.5) follows immediately from (B.4).

Appendix C

Under the risk-neutral measure, the discrete-time distribution of vector $[x_T, y_T]$ is bivariate normal with mean vector (C1) and covariance matrix (C.2):

$$\begin{bmatrix} A(0) - A(t-T) \\ q_y(t-T) \end{bmatrix} + \begin{bmatrix} B_x(t-T) & B_y(t-T) \\ 0 & Q_y(t-T) \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \qquad (C.1)$$

$$\begin{bmatrix} S_{xx}(t-T) & S_{xy}(t-T) \\ S_{xy}(t-T) & S_{yy}(t-T) \end{bmatrix}, \qquad (C.2)$$

where

$$q_y(t) \equiv m_y \frac{[Q_y(0) - Q_y(t)]}{k_y}, \qquad (C.3)$$

$$Q_y(t) \equiv exp(k_y\, t)\ (= B_x(t) - (k_x - k_y)\, B_y(t)), \qquad (C.4)$$

$$S_{xx}(t) \equiv F(0) - F(t), \qquad (C.5)$$

$$S_{xy}(t) \equiv s_y^2 \frac{[B_y(0)\, Q_y(0) - B_y(t)\, Q_y(t)]}{2k_y} \qquad (C.6)$$

$$- \frac{(s_y^2 - 2r_{xy} s_x s_y k_y)}{2k_y (k_x + k_y)}\, [B_x(0)\, Q_y(0) - B_x(t)\, Q_y(t)],$$

$$S_{yy}(t) \equiv s_y^2 \frac{[Q_y(0)^2 - Q_y(t)^2]}{2k_y}. \qquad (C.7)$$

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