Testing for Constant Hedge Ratios in Commodity Markets: A Multivariate GARCH Approach

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Abstract

We develop a new multivariate GARCH parameterization that is suitable for testing the hypothesis that the optimal futures hedge ratio is constant over time, given that the joint distribution of cash and futures prices is characterized by autoregressive conditional heteroskedasticity. The advantage of the new parameterization is that it allows for a flexible form of time-varying volatility, even under the null of a constant hedge ratio. The model is estimated using weekly corn prices. Statistical tests reject the null hypothesis of a constant hedge ratio and also reject the null that time variation in optimal hedge ratios can be explained solely by deterministic seasonality and time-to-maturity effects.

Key words: autoregressive conditional heteroskedasticity, futures, hedging

TESTING FOR CONSTANT HEDGE RATIOS IN COMMODITY MARKETS: A MULTIVARIATE GARCH APPROACH

Hedging with futures contracts is an important risk management strategy for firms dealing with commodities, the prices of which are notoriously volatile. Hedging reduces risk because cash and futures prices for the same commodity tend to move together, so that changes in the value of a cash position are offset by changes in the value of an opposite futures position. Because cash and futures price movements are typically not perfectly correlated (i.e., there is basis risk), risk management requires determination of the "optimal hedge ratio" (the optimal amount of futures bought or sold expressed as a proportion of the cash position). When basis risk is the only source of uncertainty,[1] the optimal hedge ratio often can be reduced to a simple ratio of the conditional covariance between cash and futures prices to the conditional variance of futures prices (Benninga, Eldor, and Zilcha, 1983; Myers, 1991; Lence, 1995). To estimate such a ratio, early work simply used the slope of an ordinary least squares regression of cash on futures prices. An improved procedure is possible by computing the relevant moments of the price distribution relative to the proper conditional means (Myers and Thompson, 1989; Moschini and Lapan, 1995).[2] More generally, estimation of the optimal hedge ratio recognizes that commodity cash and futures prices often display time-varying volatility and relies on techniques consistent with such a hypothesis, such as Engle's (1982) autoregressive conditional heteroskedasticity (ARCH) framework or Bollerslev's (1986) generalized ARCH (GARCH) approach.

ARCH and GARCH models appear ideally useful for estimating time-varying optimal hedge ratios, and a number of applications have concluded that such ratios seem to display considerable variability over time (Cecchetti, Cumby, and Figlewski, 1988; Baillie and Myers, 1991; Myers, 1991; Kroner and Sultan, 1993). Yet, no existing study has provided compelling evidence that such time-varying hedge ratios are statistically different from a constant hedge ratio. A time-varying covariance matrix of cash and futures prices, per se, is not sufficient to establish that the optimal hedge ratio is time varying. Constancy of the hedge ratio restricts the ratio of the covariance between cash and futures prices to the variance of futures prices to be constant, but it need not restrict the moments of the joint distribution of cash and futures prices in any other way. Unfortunately, the particular parametric GARCH models that have been used to date admit a constant hedge ratio only under very restrictive conditions, so that the hypothesis of a constant optimal hedge ratio can be tested only jointly with other hypotheses. The main purpose of this paper is to develop a more general GARCH parameterization that yields a constant hedge ratio as a special case, while still allowing for a flexible time-varying distribution of cash and futures prices. The model is illustrated with an application to the problem of storage hedging of corn using futures prices from the Chicago Board of Trade and Iowa cash prices for the period 1976-1997.

The Optimal Futures Hedge and GARCH Models

A typical hedging model in our setting involves a decision maker who allocates wealth between a risk-free asset and two risky assets: the physical commodity and the corresponding futures (Myers, 1991). Let y_t^* and z_t^* denote the optimal quantity of the physical commodity bought and futures sold, respectively, with both positions taken at time *t-1* and held until time *t*. The optimal hedge ratio (*OHR*) is defined as $OHR_t \equiv z_t^*/y_t^*$. Assumptions about preferences and/or the distribution of cash and futures prices are typically necessary to characterize this ratio. But a useful result obtains when the futures price f_t and cash price p_t are conditionally jointly normally distributed and the futures market is unbiased. In such a case $OHR_t = \dfrac{Cov(p_t,f_t|\Omega_{t-1})}{Var(f_t|\Omega_{t-1})}$, where Ω is the information set, implying that the hedge ratio of interest is independent of risk preferences.[3] If the joint distribution of cash and futures prices changes over time, then OHR_t as defined above may also change over time.[4] The time path of OHR_t can be calculated given knowledge of the (time-dependent) covariance matrix for cash and futures prices, which can be estimated with GARCH models. But, clearly, the optimal hedge ratio can still be constant even if $Var(f_t|\Omega_{t-1})$ and $Cov(p_t,f_t|\Omega_{t-1})$ both vary over time, as long as the covariance term is proportional to the variance term (i.e., $Cov(f_t|\Omega_{t-1}) = \gamma_0 \; Var(f_t|\Omega_{t-1})$ for all t , for some constant γ_0). Thus, although a constant hedge ratio is restrictive, it is a legitimate possibility, even with time-varying conditional price distributions.

Because a constant hedge ratio would simplify implementation of an optimal hedging strategy, it is of considerable interest to test this hypothesis. But to do that, a new GARCH specification is desirable because existing GARCH models parameterizations are too restrictive. To illustrate, write the conditional mean of cash and futures prices as $p_t = E[p_t|\Omega_{t-1}] + u_{1,t}$ and $f_t = E[f_t|\Omega_{t-1}] + u_{2,t}$, respectively. The conditional covariance matrix of cash and futures prices can then be written as $H_t \equiv E[u_t u_t{'}|\Omega_{t-1}]$, where $u_t \equiv [u_{1,t}, u_{2,t}]{'}$ is the vector of innovations in the cash and futures prices. Once the parameters of the conditional covariance matrix are estimated, say by a GARCH model, the optimal hedge ratio is computed as $OHR_t = h_{12,t}/h_{22,t}$, where $h_{ij,t}$ denotes the $(i,j)th$ element of H_t. Several alternative multivariate GARCH (MGARCH) parameterizations of the conditional covariance matrix H_t have been used in this setting, but all have shortcomings for testing the null hypothesis of a constant optimal hedge ratio. Consider, for instance, the constant conditional correlation model used by Cecchetti, Cumby, and Figlewski (1988) to estimate time-varying hedge ratios within an ARCH framework, or the constant conditional correlation GARCH model developed by Bollerslev (1990) and applied to hedge ratio estimation by Kroner and Sultan (1993). This specification is elegant and computationally attractive. However, with a constant conditional correlation coefficient ρ, the optimal hedge ratio must satisfy $OHR_t = \rho \sqrt{h_{11,t}/h_{22,t}}$. Thus, in this specification a constant hedge ratio can be obtained only if the variance of the cash price is perfectly proportional to the variance of the futures price, a condition that is patently unattractive.

Other popular MGARCH parameterizations also are not suited to testing for a constant optimal hedge ratio. The “diagonal vech” specification (Baillie and Myers, 1991; Myers, 1991), for example, admits a constant optimal hedge ratio only when there are no GARCH effects (i.e., it can only arise if the conditional covariance matrix is itself constant). Furthermore, this specification does not restrict H_t to be positive definite (PD), which turns out to be troublesome for estimation of

MGARCH models of cash and futures prices.[5] The "general vech" parameterization can improve on the flexibility of the model for testing a constant OHR, but this model is usually over-parameterized and difficult to estimate because it does not require H_t to be PD either. The PD MGARCH specification estimated by Baillie and Myers (1991) overcomes this latter problem. But a constant optimal hedge ratio in this model requires that the correlation between cash and futures be restricted to equal unity (Moschini and Aradhyula, 1993), implying that there is no basis risk (cash and futures prices are perfectly correlated) and thus no meaningful hedging problem.

A New GARCH Parameterization for OHR Estimation and Testing

To overcome the limitations of existing GARCH parameterizations for optimal hedge ratio estimation and testing, we develop a new specification that is rooted in what Engle and Kroner (1995) have termed the "BEKK" parameterization. The "modified BEKK" parameterization that we propose is defined as[6]

$$\begin{aligned} H_t &= \Gamma_t' [C_t' C_t + \sum_{k=1}^{K} \sum_{i=1}^{q} A_{ik}' u_{t-i} u_{t-i}' A_{ik} + \sum_{k=1}^{K} \sum_{j=1}^{r} G_{jk}' H_{t-j} G_{jk}] \Gamma_t \\ &\equiv \Gamma_t' \tilde{H}_t \Gamma_t \end{aligned} \tag{1}$$

where Γ_t is a 2×2 matrix of (possibly time-varying) parameters to be defined below, A_{ik} and G_{jk} are 2×2 parameter matrices, and C_t is a (time-varying) 2×2 upper triangular matrix that depends on a vector x_t of weakly exogenous variables. Note that, if Γ_t were the identity matrix, our parameterization would reduce to a conventional BEKK model. When K=1 the BEKK parameterization is just a PD MGARCH model, and for arbitrary K this model can be made quite general. Indeed, Engle and Kroner (1995) show that for an appropriate choice of K, the BEKK parameterization can be made fully general (i.e., it is equivalent to any PD general vech parameterization). When Γ_t is not the identity matrix our specification can be a useful generalization of the BEKK model, and as long as Γ_t is a PD matrix our modified BEKK model will maintain the PD property. This fact is an important advantage in numerical optimization (particularly for cash-futures price models, as mentioned in endnote 5).

To make this model operational, restrictions on the A_{ik} and G_{jk} matrices are required to ensure identification and to cut down on the over-parameterization that typically characterizes the BEKK model. Here we set $K = 2$ and define the 2×2 A_{ik} and G_{jk} parameter matrices as

$$A_{i1} = \begin{bmatrix} a_{11,i1} & 0 \\ 0 & a_{22,i1} \end{bmatrix} , \quad A_{i2} = \begin{bmatrix} a_{11,i2} & 0 \\ 0 & 0 \end{bmatrix}$$
$$G_{j1} = \begin{bmatrix} g_{11,j1} & 0 \\ 0 & g_{22,j1} \end{bmatrix} , \quad G_{j2} = \begin{bmatrix} g_{11,j2} & 0 \\ 0 & 0 \end{bmatrix} . \tag{2}$$

Engle and Kroner (1995) have shown that in the general BEKK model any bivariate PD diagonal vech parameterization can be represented by a unique BEKK parameterization with these restrictions on the A_{ik} and G_{jk} matrices. Furthermore, we define the (non-zero) *ij*th elements of the upper triangular matrix C_t as $c_{ij,t} = x_t^{/}\delta_{ij}$, where the first element of x_t is a constant, and δ_{ij} is a J×1 parameter vector.[7] Including the exogenous variables in this way does not restrict the sign of the impact that they can have on volatility levels, a desirable property in our context given that x_t will often include deterministic variables such as seasonality and time-to-maturity effects.

Finally, to generalize the BEKK model to make it useful for testing the constant optimal hedge ratio hypothesis, we write the Γ_t matrix as

$$\Gamma_t = \begin{bmatrix} 1 & 0 \\ \gamma_0 & 1 \end{bmatrix} , \quad \forall \ t \tag{3}$$

where γ_0 is a constant parameter to be estimated. Given the parameterizations in (1) and (3), the covariance matrix H_t can be written as

$$H_t = \begin{bmatrix} \tilde{h}_{11,t} + 2\gamma_0\tilde{h}_{12,t} + \gamma_0^2\tilde{h}_{22,t} & \tilde{h}_{12,t} + \gamma_0\tilde{h}_{22,t} \\ \tilde{h}_{12,t} + \gamma_0\tilde{h}_{22,t} & \tilde{h}_{22,t} \end{bmatrix} \tag{4}$$

where $\tilde{h}_{ij,t}$ is the *ij*th element of $\tilde{H}_t$. Notice that if $\tilde{h}_{12,t} = 0 \;\; \forall\; t$, then $OHR_t = \gamma_0$, while the conditional variance equations for $h_{11,t}$ and $h_{22,t}$ remain relatively unrestricted. Hence, our modified BEKK parameterization provides a reasonably parsimonious model that is easy to estimate (PD imposed) and flexible enough to allow time-varying hedge ratios but that facilitates a simple and meaningful test of the constant optimal hedge ratio hypothesis (i.e., $\tilde{h}_{12,t} = 0,\; \forall\; t$). From equations (1) and (2), the elements of the $\tilde{H}_t$ matrix can be written as

$$\begin{aligned}
\tilde{h}_{11,t} &= c_{11,t}^2 + \sum_{i=1}^{q} (a_{11,i1}^2 + a_{11,i2}^2)\, u_{1,t-i}^2 + \sum_{j=1}^{r} (g_{11,j1}^2 + g_{11,j2}^2)\, h_{11,t-j} \\
\tilde{h}_{12,t} &= c_{11,t}\, c_{12,t} + \sum_{i=1}^{q} a_{11,i1}\, a_{22,i1}\, u_{1,t-i}\, u_{2,t-i} + \sum_{j=1}^{r} g_{11,j1}\, g_{22,j1}\, h_{12,t-j} \\
\tilde{h}_{22,t} &= c_{12,t}^2 + c_{22,t}^2 + \sum_{i=1}^{q} a_{22,i1}^2\, u_{2,t-i}^2 + \sum_{j=1}^{r} g_{22,j1}^2\, h_{22,t-j}\,.
\end{aligned} \tag{5}$$

Hence, a set of parametric restrictions sufficient to ensure $\tilde{h}_{12,t} = 0,\; \forall\; t$, is $a_{11,i1} = g_{11,j1} = 0$, $\forall\; (i,j)$, and $\delta_{12} = 0$ (this last set of restrictions ensures $c_{12,t} = 0\;,\; \forall\; t$).[8]

It is possible to generalize our model to obtain a test for the hypothesis that optimal hedge ratios vary only with exogenous variables x_t but are otherwise not time dependent (i.e., not affected by lagged $u_t u_t^{/}$ and lagged H_t). Specifically, write Γ_t as

$$\Gamma_t = \begin{bmatrix} 1 & 0 \\ x_t^{/}\gamma & 1 \end{bmatrix} \tag{6}$$

where γ is a $(J \times 1)$ vector of parameters on x_t. Then the conditional covariance matrix becomes

$$H_t = \begin{bmatrix} \tilde{h}_{11,t} + 2(x_t^{/}\gamma)\tilde{h}_{12,t} + (x_t^{/}\gamma)^2\tilde{h}_{22,t} & \tilde{h}_{12,t} + (x_t^{/}\gamma)\tilde{h}_{22,t} \\ \tilde{h}_{12,t} + (x_t^{/}\gamma)\tilde{h}_{22,t} & \tilde{h}_{22,t} \end{bmatrix}. \tag{7}$$

Hence, if $\tilde{h}_{12,t} = 0 \;\; \forall\; t$, then $OHR_t = x_t^{/}\gamma$ and, in this case, the optimal hedge ratio only changes over time with changes in x_t.

In conclusion, our modified BEKK model not only facilitates estimation (by imposing the PD restriction) but also allows for a simple, informative test of the constant (or varying only with x_t) optimal hedge ratio hypothesis. For both specifications of the Γ_t matrix that we considered, the null hypothesis entails the following $(q + r + J)$ parametric restrictions:

$$\begin{aligned} a_{11,i1} &= 0 \quad , \quad \forall\ i = 1, 2, \ldots, q \\ g_{11,j1} &= 0 \quad , \quad \forall\ j = 1, 2, \ldots, r \\ \delta_{12} &= 0. \end{aligned} \tag{8}$$

If the null hypothesis is not rejected, then either γ_0 or $x_t'\gamma$ is the optimal hedge ratio (depending on the specification of Γ_t). If the null is rejected, then $OHR_t = h_{12,t} / h_{22,t}$, where $h_{ij,t}$ denote the unrestricted conditional moments from (4) or (7). Finally, if Γ_t is the identity matrix, then the model reverts to the conventional BEKK parameterization. All of these restrictions can be tested easily using a likelihood ratio or Wald approach.

An Application to Hedging Corn

The modified BEKK model is applied to the problem of estimating and testing optimal hedge ratios for speculative storage of corn in the Midwest. It is assumed that an investor buys and stores corn for resale at a later period, the price of which is unknown at the time of purchase. The investor can hedge the long cash position by selling futures, has a weekly time horizon, and always uses the nearby contract (the contract with the nearest maturity date lying beyond the current month). The nearby contract is typically the most actively traded, and this liquidity makes it attractive to potential hedgers. It is assumed that the investor takes out futures positions and holds the position for a week. At the end of the week, the investor reevaluates the futures position and chooses a new hedge ratio for the following week. Hence, the hedge ratio may be adjusted every week to reflect time-varying volatility. On dates when the next week lies in a delivery month for the nearby future, it is assumed that the investor switches to hedging in the next delivery month (the “nearby” contract switches to the next delivery month).[9]

The price series used are mid-week (Thursday) prices. Cash prices are the average corn cash prices quoted in North-Central Iowa. Futures prices are Thursday closing prices for the nearby (as defined earlier) corn contracts quoted on the Chicago Board of Trade. The sample period extends from January 1976 through June 1997, with a total of 1,120 weekly observations.

Conditional Means of Cash and Futures Prices. Estimation of hedge ratios requires first specifying a model for the conditional means of cash and futures prices. We begin by noting that futures prices at different points in time for contracts with the same maturity date are clearly likely to be I(1) because of arbitrage considerations (if they were mean-reverting we could always generate profitable trading rules).

Similarly, theoretical and empirical considerations suggest that cash and futures prices for the same commodity should be closely related because transportation and storage activities ensure spatial and temporal arbitrage. To be sure, such arguments should be qualified when applied to a data set such as ours. When a new harvest intervenes between the delivery date *T* associated with the futures contract and the date *t* at which both cash and futures prices are quoted, the arbitrage conditions hold as inequalities and no exact relation between cash and futures price can be postulated. To account for these considerations in a reasonable manner, we write the conditional mean equations for cash and futures prices in dynamic form as

$$\Delta p_t = \beta_0 + \beta_1 (T - t) + \sum_{i=2}^{4} \beta_i D_{it} + \beta_5 p_{t-1} + \beta_6 f_{t-1}(T) + u_{1,t} \qquad (9)$$

$$\Delta f_t(T) = \mu + u_{2,t} \qquad (10)$$

where $f_t(T)$ is futures price at *t* for delivery at the nearby expiration date *T*, p_t is cash price at *t*, the D_{it} are quarterly dummy variables, and $u_{1,t}$ and $u_{2,t}$ are random shocks.[10]

The inclusion of the intercept μ in (10) is motivated primarily by econometric considerations, to ensure that the estimated residuals $u_{2,t}$ have a zero mean. The parameter β_0, on the other hand, captures the fact that (other things being equal) the cash price rises throughout the crop year to reflect carrying charges (storage costs). However, because we are using data spanning many crop years, we have to allow for the fact that the rate of growth in cash prices may vary seasonally, and that the cash price normally drops around harvest periods, hence the inclusion of the seasonal dummy variables D_{it}.

Equations (9) and (10) are consistent with cash and futures prices being I(1) and cointegrated (Enders, 1995), but they allow the long-run equilibrium relation between cash and futures prices to be influenced by seasonality D_{it} and by time to maturity $(T\text{-}t)$. Time to maturity is included in the equilibrium relationship to reflect the fact that we expect cash and futures prices to converge at maturity (other things being equal, the cost of storage should cause a larger difference between cash and futures prices the further the futures are from maturity).

Estimation. In the present application the innovations $u_t \equiv [u_{1,t},\ u_{2,t}]'$ are assumed to follow a bivariate GARCH(1,1) process, which is parameterized by the modified BEKK model discussed earlier with $K=2$, $q=1$, and $r=1$. Estimation is carried out using quasi-maximum likelihood methods that employ a conditional normal distribution for u_t. Many studies have found commodity price innovations to have fatter tails than normal (e.g., Baillie and Myers, 1991). For this reason, we test the normality assumption and use quasi-maximum likelihood standard errors for hypothesis testing (Bollerslev and Wooldridge, 1992; Lumsdaine, 1996). There are I(1) variables in equation (9) but these variables are assumed to form a stationary linear combination (long-run equilibrium), and so maximum likelihood estimation, and any hypothesis tests done on the conditional variance part of the model, is valid (Sims, Stock, and Watson, 1990; Phillips, 1991).

Results. Results for the general modified BEKK are shown in Table 1, along with results when the deterministic hedge ratio restrictions (hedge ratios only vary with the deterministic seasonal dummy and time-to-maturity variables) and constant hedge ratio restrictions are imposed. The "t-ratios" in the table are a ratio of the estimated parameter to the standard error estimated via quasi-maximum likelihood. The deterministic component of the conditional variance part of each of these models has a constant $\delta_{ij,0}$, a

TABLE 1. Estimated GARCH models for corn

Parameter	General Modified BEKK		Deterministic Hedge Ratio Model		Constant Hedge Ratio Model	
	Estimate	t-ratio	Estimate	t-ratio	Estimate	t-ratio
Conditional Mean						
μ	-0.1598	-0.85	-0.1324	-0.71	-0.1069	-0.58
β_0	0.5987	0.94	1.2462	1.75	1.1789	1.65
β_1	-0.0069	-1.59	-0.0147	-3.17	-0.0141	-3.01
β_2	-0.4938	-2.57	-0.4217	-1.98	-0.4083	-1.92
β_3	-1.4814	-5.25	-1.7112	-5.14	-1.7827	-5.25
β_4	-0.3010	-1.10	-0.3052	-1.04	-0.3098	-1.08
β_5	-0.0523	-4.00	-0.0599	-5.32	-0.0623	-5.01
β_6	0.0485	4.05	0.0543	5.23	0.0566	4.98
Conditional Variance						
$\delta_{11,0}$	-0.6563	-0.97	-1.0959	-1.58	1.1307	1.87
$\delta_{11,1}$	-0.0154	-1.37	-0.0180	-1.61	0.0157	1.57
$\delta_{11,2}$	2.9748	4.49	1.5014	1.63	-1.5430	-2.07
$\delta_{11,3}$	-0.7121	-0.75	-1.0029	-1.65	1.1831	2.03
$\delta_{11,4}$	0.1250	0.13	0.0480	0.10	-0.0360	-0.07
$\delta_{12,0}$	-1.2602	-1.55				
$\delta_{12,1}$	0.0169	1.21				
$\delta_{12,2}$	2.3719	2.27				
$\delta_{12,3}$	0.0972	0.07				
$\delta_{12,4}$	0.3505	0.27				
$\delta_{22,0}$	1.2669	1.83	-0.3540	-0.40	-2.2122	-2.31
$\delta_{22,1}$	-0.0046	-0.37	-0.0122	-0.96	0.0251	1.02
$\delta_{22,2}$	0.5667	1.24	3.2365	5.02	-1.1996	-2.97
$\delta_{22,3}$	-2.6184	-2.78	2.8192	2.87	2.0790	1.35
$\delta_{22,4}$	-2.5128	-3.59	2.3072	2.72	-0.2358	-0.31
$a_{11,11}$	0.3775	8.42				
$a_{22,11}$	0.4312	8.03	0.4036	7.17	0.4017	7.08
$a_{11,12}$	0.1219	2.70	0.2196	5.01	0.2204	4.88
$g_{11,11}$	0.6943	9.31				
$g_{22,11}$	0.8723	27.67	0.8879	27.50	0.8915	28.30
$g_{11,12}$	0.0448	0.26	0.3358	7.19	0.3448	8.28
γ_0	0.0914	1.12	0.8855	13.70	0.8845	44.39
γ_1	0.0019	2.19	-0.0011	-1.22		
γ_2	0.0326	0.73	0.0488	0.82		
γ_3	0.0227	0.45	0.1367	1.90		
γ_4	0.0131	0.20	0.0730	0.96		
Log-Likelihood	-6529.71		-6585.63		-6589.27	

time-to-maturity parameter $\delta_{ij,1}$ (time to maturity is measured in days), and quarterly dummy variable parameters $\delta_{ij,2}$, $\delta_{ij,3}$ and $\delta_{ij,4}$ (for quarters 2-4). The quarterly dummy variables represent seasonal variation in the conditional variance while the time-to-maturity variable accounts for the fact that different futures prices may behave differently depending on time to maturity. The time-to-maturity variable also accounts for a possible "jump" in volatility when shifting from a maturing futures contract to the next nearby maturity date. These same variables are used to investigate deterministic movements in the hedge ratio over time [i.e., they are used as the x_t variables in (6)].

We see from the conditional mean estimates at the top of Table 1 that cash price movements have statistically significant seasonality at conventional significance levels, and that cash prices, futures prices, and time to maturity are related in an intuitive way (for example, an increase in time to maturity increases the difference between the current futures and cash price).[11] In the conditional variance part of the models there appears to be significant seasonality and time-to-maturity effects (as expected), as well as significant conditional heteroskedasticity. It is also interesting to note that in all three models reported in Table 1, at least some of the Γ_t parameters are statistically different from zero at conventional significance levels.

Table 2 contains model evaluation statistics for the three models (general modified BEKK, deterministic hedge ratio, and constant hedge ratio). We see that the cash price residuals from the general modified BEKK do not appear to be autocorrelated, and the conditional variance model does a good job of explaining cash price conditional heteroskedasticity. In the futures price equation there is weak evidence of high-order autocorrelation in the residuals and in the squared standardized residuals (suggesting the possibility of residual GARCH effects in the futures price equation not captured by the model). Overall, however, the general modified BEKK seems to fit the data reasonably well. In the models incorporating the deterministic hedge ratio restrictions and constant hedge ratio restrictions, we see that there is very strong evidence of remaining conditional heteroskedasticity in both the cash and futures price equations. This immediately suggests that the general modified BEKK provides a better fit and that the deterministically varying and constant hedge ratio models may not be consistent with these data. Sample

TABLE 2. Model evaluation statistics

Test	General Modified BEKK		Deterministic Hedge Ratio Model		Constant Hedge Ratio Model	
	Statistic	**p-value**	**Statistic**	**p-value**	**Statistic**	**p-value**
Corn						
Cash:						
$Q(1)$	0.016	0.899	0.001	0.976	0.002	0.967
$Q(5)$	3.381	0.641	3.439	0.633	3.350	0.646
$Q(10)$	11.560	0.316	11.390	0.328	11.222	0.341
$Q^2(1)$	0.726	0.394	4.671	0.031	4.406	0.036
$Q^2(5)$	1.405	0.924	12.684	0.027	17.321	0.004
$Q^2(10)$	5.822	0.830	19.846	0.031	23.854	0.008
$\hat{K}$						
JB	-0.402		-0.358		-0.397	
	3.847		3.797		3.874	
	63.478	0.000	53.604	0.000	64.977	0.000
Futures:						
$Q(1)$	0.435	0.510	0.435	0.510	0.435	0.510
$Q(5)$	10.171	0.071	10.171	0.071	10.171	0.071
$Q(10)$	18.855	0.042	18.855	0.042	18.855	0.042
$Q^2(1)$	0.870	0.351	2.136	0.144	1.717	0.190
$Q^2(5)$	11.783	0.038	14.717	0.012	15.957	0.007
$Q^2(10)$	18.061	0.054	20.320	0.026	21.557	0.018
$\hat{S}$	0.024		0.059		0.055	
$\hat{K}$	3.648		3.861		3.882	
JB	19.669	0.000	35.211	0.000	36.862	0.000

Note: $Q(df)$ is the Ljung-Box Q statistic for testing *df* degree autocorrelation in the residuals; $Q^2(df)$ is the corresponding statistic for testing *df* degree autocorrelation in the squared standardized residuals; $\hat{S}$ is the sample skewness of the standardized residuals; $\hat{K}$ is the sample kurtosis of the standardized residuals; and *JB* is the Jarque-Bera test for normality of the standardized residuals.

skewness and kurtosis of the standardized residuals differs from what we would expect under normality, and the Jarque-Bera test rejects normality at essentially any significance level in all three models.

Because of the rejection of conditional normality we use Wald tests based on the quasi-maximum likelihood covariance matrix to compare various specializations of the model to the general modified BEKK. These Wald test results are reported in Table 3. The first pair of tests investigate the restrictions that the covariance matrix is constant (constant covariance matrix), and that the covariance matrix changes only with deterministic seasonal and time-to-maturity effects (deterministic covariance matrix). Both of these sets of restrictions are soundly rejected against the general modified BEKK. Hence, there is strong evidence that the conditional covariance matrix of cash and futures prices does vary through time, and that this variation cannot be explained simply by seasonal and/or time-to-maturity effects.

The second pair of tests in Table 3 investigate whether a conventional BEKK model (with and without deterministic components) can explain these data as well as the general modified BEKK (i.e., we test the restriction that Γ_t is the identity matrix for all t). Both conventional BEKK specifications are also rejected against the general modified BEKK. This suggests that the general modified BEKK may be a useful parameterization in its own right, quite apart from its uses as a meaningful model to test the constant hedge ratio hypothesis.

The third pair of tests in Table 3 investigate whether the deterministic hedge ratio and constant hedge ratio restrictions given in (8) above are formally rejected against the general modified BEKK alternative. In both cases the null hypothesis is rejected at essentially any significance level. This provides strong evidence that optimal hedge ratios are indeed time varying and in ways that cannot be explained simply by deterministic seasonal and time-to-maturity effects. It appears that, in this application, hedge ratios do vary over time in ways that can be captured using GARCH models.

To investigate how much time variation is occurring in the estimated optimal hedge ratios, we graph the in-sample modified BEKK hedge ratios in Figure 1, together with the constant hedge ratio estimate. In Figure 2 we graph the deterministically varying hedge ratios (seasonal and time-to-maturity effects only), again together with the constant hedge ratio estimate. It is clear that the modified BEKK hedge ratio displays considerable additional time variation compared to either the constant hedge ratio or a hedge ratio

TABLE 3. Quasi-maximum likelihood Wald tests

Model	Number of Parameters	Wald χ^2 Statistic	p-Value
General modified BEKK	34		
Constant covariance matrix	11	43,896.6	0.0000
Deterministic covariance matrix	23	7,945.6	0.0000
Conventional BEKK (without deterministic component)	17	117.7	0.0000
Conventional BEKK (with deterministic component)	29	12.3	0.0306
Deterministic hedge ratio	27	317.7	0.0000
Constant hedge ratio	23	372.7	0.0000

that only is allowed to vary deterministically with seasonal and time-to-maturity effects. This indicates that, as well as being statistically significant, the time-varying optimal hedge ratios estimated with the modified BEKK can lead to very different hedging decisions compared to the alternative constant or deterministically varying hedge ratio assumptions.

Conclusion

In this paper we have provided a new GARCH parameterization that modifies the Engle and Kroner (1995) BEKK formulation. The new parameterization is particularly useful for estimating time-varying optimal hedge ratios and testing the null hypothesis that they are constant over time. Our approach overcomes an important limitation of previous studies, where the null hypothesis of a constant hedge ratio was only identified jointly with other restrictive conditions (such as, for example, that the distribution of cash and futures prices is time-invariant). As shown in this paper, such additional restrictive conditions are not necessary to obtain a constant optimal hedge ratio. In particular, we have developed modified BEKK parameterizations for the bivariate GARCH(q,r) model that nest the hypothesis of a constant hedge ratio (or of an exogenously varying hedge ratio) but retain flexible time-varying variances and covariances, even under the null hypothesis.

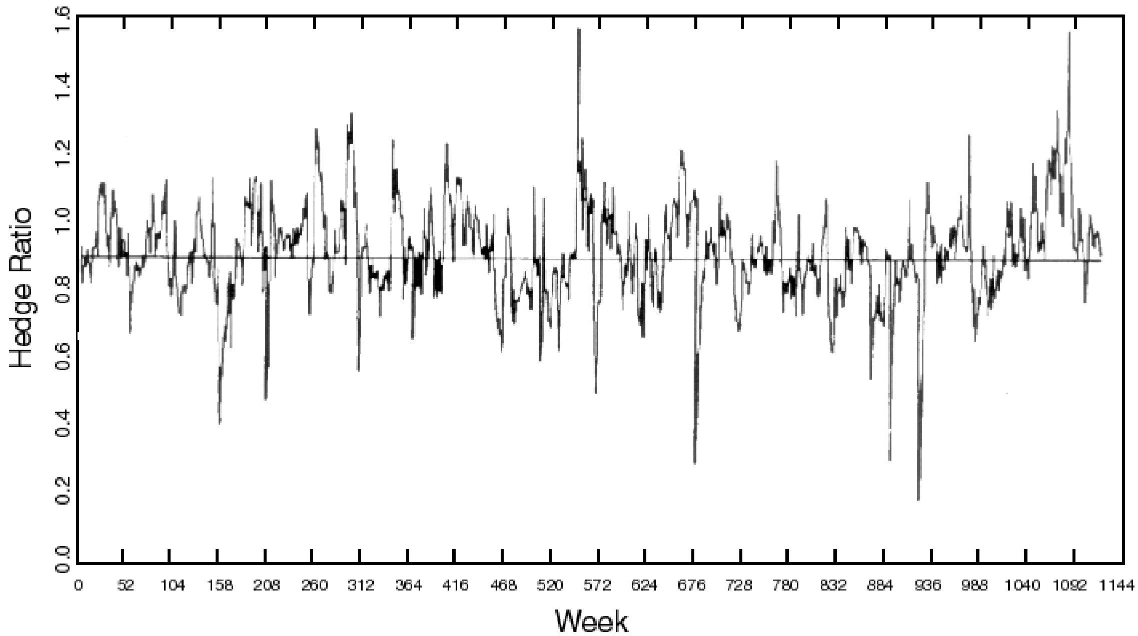

FIGURE 1. Stochastic time-varying hedge ratios for corn, January 1976 through June 1997

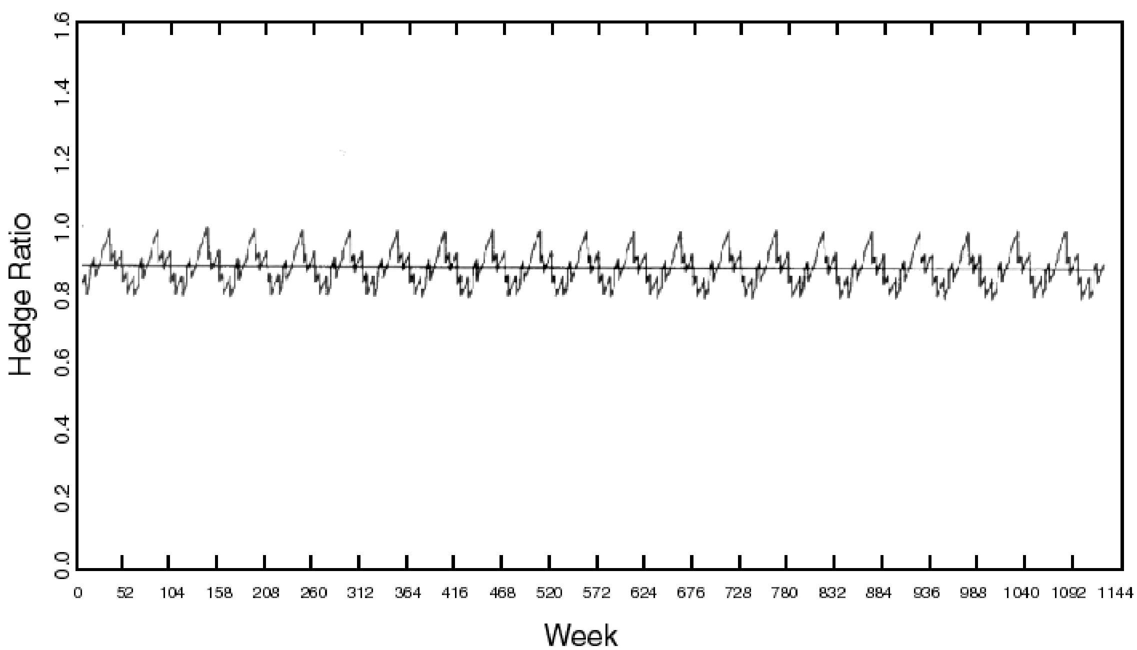

FIGURE 2. Deterministic time-varying hedge ratios for corn, January 1976 through June 1997

These modified BEKK parameterizations were utilized to estimate bivariate GARCH models for corn cash and futures prices, based on weekly data spanning the period 1976 to 1997. We find significant GARCH effects in the cash and futures prices, and these GARCH effects are still present even when accounting separately for seasonality and time to maturity, which are themselves significant components of the time variation in the covariance matrix. Furthermore, using Wald quasi-maximum likelihood tests we formally reject the null hypothesis that the ratio of conditional covariance of futures and cash prices to conditional variance of futures prices (the optimal hedge ratio) is constant at essentially any significance level. We also reject the null hypothesis that optimal hedge ratios vary only systematically with seasonality and time-to-maturity effects at essentially any significance level. Thus, our statistical tests support the conclusion that optimal hedge ratios for weekly storage hedging of corn in the Midwest are indeed time varying in ways that cannot be explained simply by seasonality and time-to-maturity effects.

Endnotes

1. This setting applies mostly to commodity handlers (country elevators, shippers, millers, storers, etc.) but not to producers because the latter are typically also exposed to quantity (production) uncertainty. For a review of more general hedging problems see Moschini and Hennessy (2001).

2. If cash and futures prices follow a martingale process, for instance, the slope of a regression of cash price changes on futures price changes (and not the slope of a regression in levels) estimates the relevant hedge ratio.

3. This useful result can actually be obtained under slightly weaker conditions than normality (Benninga, Eldor, and Zilcha, 1983; Moschini, Lapan, and Hanson, 1991; Lence, 1995).

4. This time variation should be carefully distinguished from the revision of the hedge ratio that may be due to the gradual resolution of uncertainty, as in Anderson and Danthine (1983) and Karp (1988).

5. Cash and futures prices for the same commodity tend to move closely together and have similar variances, so their H_t is almost singular. Thus, maximum likelihood estimation of H_t often iterates into the parameter range where $|H_t| \leq 0$, at which point estimation breaks down (under normality). Whereas there are various programming approaches that can help overcome this problem, it is preferable to impose the PD restriction a priori whenever possible.

6. In keeping with our hedging model, here we present our modified BEKK model for the bivariate case. Extension to the n-variate case is straightforward and is left to the interested reader.

7. McNew and Fackler (1994) model time-varying hedge ratios with such a structural specification of the covariance matrix but do not embed that in an ARCH or GARCH model.

8. As apparent from (5), we could substitute $a_{22,i1} = g_{22,j1} = 0$ for $a_{11,i1} = g_{11,j1} = 0$ but this would destroy the flexibility of the model ($\tilde{h}_{22,t}$ is restricted to equal just $c_{22,t}^2$ under the null of a constant OHR).

9. There are five delivery months for Chicago Board of Trade corn: March, May, July, September, and December.

10. Because we are chaining quotes from several contracts by using nearby futures prices, it is important to ensure that the difference $\Delta f_t(T)$ is computed from two contiguous observations of the same contract. Also, weekly dummy variables could have been used instead of quarterly, but empirical results suggested that quarterly dummy variables are sufficient to capture the major seasonal patterns in growth rates for corn cash prices in our data.

11. This is indicated by the negative signs on the estimated β_1 and β_5, together with the positive sign on the estimate of β_6.

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